Numerical Simulations of a Possible Hypercomputational Quantum Algorithm

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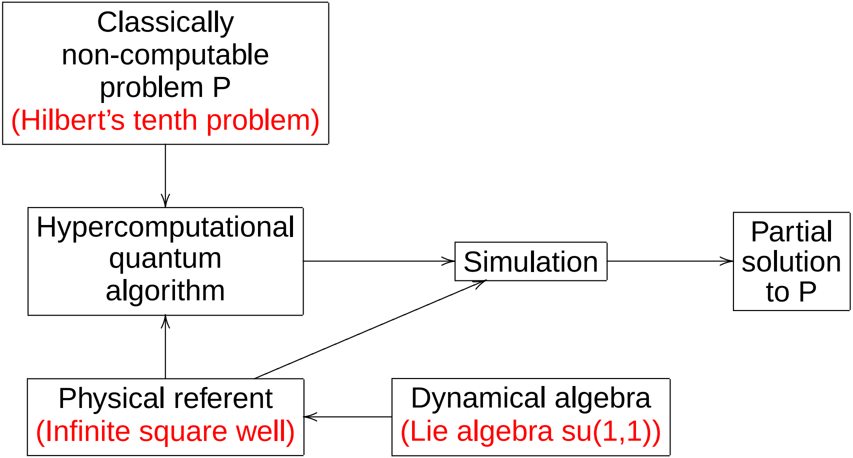
Key ideas



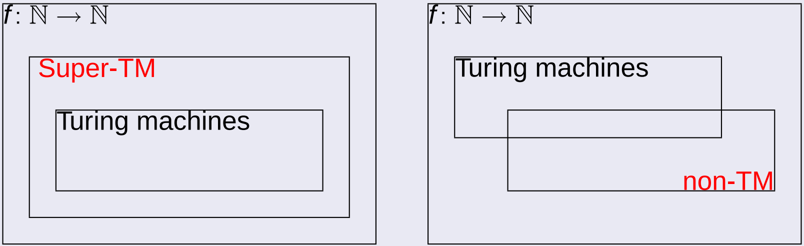
Classically non-computable problem P
(Hilbert's tenth problem)
Hypercomputational quantum algorithm
Simulation
Partial solution to P
Physical referent
(Infinite square well)
Dynamical algebra
(Lie algebra su(1,1))

Hypercomputation I

Definition

A hypercomputer is any machine (theoretical or real) that compute functions or numbers, or more generally solve problems or carry out tasks, that cannot be computed or solved by a Turing machine (TM)

Types



Hypercomputation II

Examples

- Oracle Turing Machine (Turing)
- Accelerating Turing machine (Copeland)
- Analog Recurrent Neural Network (Sielgelmann and Stong)
- ⋮

Implementation

The possibility of real construction of a hypercomputer is controversial and is still under analysis.

Incomputable-(Turing Machine) problem

Hilbert's tenth problem

Given a Diophantine equation

$$D(x_1, \ldots, x_k) = 0 \ ,$$

we should build a procedure to determine whether or not this equation has a solution in $\mathbb{N}$.

Classical solution (Matiyasevich, Davis, Robinson, Putnam)

Hilbert's tenth problem	$\equiv$	Halting problem for Turing Machine

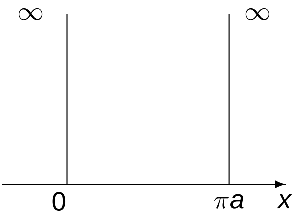
Physical referent: Infinite Square Well I

- Quantum system: particle with mass m trapped inside the infinite square well $0 \leq x \leq \pi a$

$$V(x) = \begin{cases} 0 \equiv \dfrac{\hbar^2}{2ma^2}, & \text{if } x \in (0, \pi a) \ , \\ \infty, & \text{otherwise} \ , \end{cases}$$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{2ma^2} \ ,$$

$$\psi(x) = 0, \quad x \geq \pi a \text{ and } x \leq 0 \ ,$$



Physical referent: Infinite Square Well II

- Computational basis and action of H on it:

$$\{|n\rangle \mid n \in \mathbb{N}\} \quad ,$$

$$H\,|n\rangle = E_n\,|n\rangle\,, \text{ where } E_n = (\hbar^2/2ma^2)n(n+2) \quad .$$

- Dynamical Lie algebra $\mathfrak{su}(1,1)$ associated with ISW:

$$[K_-, K_+] = K_3 \quad , \qquad [K_\pm, K_3] = \mp 2K_\pm \quad .$$

- Infinite-dimensional irreducible representation for $\mathfrak{su}(1,1)$

$$K_+\,|n\rangle = \sqrt{(n+1)(n+3)}\,|n+1\rangle \text{ (creation operator)} \quad ,$$

$$K_-\,|n\rangle = \sqrt{n(n+2)}\,|n-1\rangle \text{ (annihilation operator)} \quad ,$$

$$K_3\,|n\rangle = (2n+3)\,|n\rangle \text{ (Cartan operator)} \quad .$$

Physical referent: Infinite Square Well III

- Number operator

$$N = (1/2)(K_3 - 3) = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} ,$$

$$N \,|\, n\rangle = n \,|\, n\rangle \quad .$$

- Barut-Girardello coherent states ($K_- \,|\, z\rangle = z \,|\, z\rangle$, with $z \in \mathbb{C}$):

$$|\, z\rangle = \frac{|z|}{\sqrt{I_2(2\,|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(n+2)!}} \,|\, n\rangle \quad .$$

Algorithm's strategics I

Codification *à la* Kieu

$$
\begin{array}{ccc}
D(x_1,\ldots,x_k)=0 & \xrightarrow{\text{codification}} & H_{\rm D}=(D(N_1,\ldots,N_k))^2 \\
\downarrow & & \downarrow \\
\text{Solution in } \mathbb{N} & \xrightarrow{\text{if and only if}} & H_{\rm D}\,|\,\{n\}\rangle_0=0
\end{array}
$$

Note: Continuos quantum computation

New problem

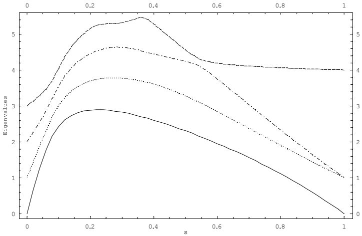
To find the ground state $|\,\{n\}\rangle_0$ of $H_{\rm D}$.

Algorithm's strategics II

Solution *à la* Kieu: adiabatic quantum computation

$$H_{\rm A}(t) = (1 - t/T)H_{\rm I} + (t/T)H_{\rm D} \text{ over } t \in [0, T]$$

$$H_{\rm I} = \sum_{i=1}^{k} (K_{+_i} - z_i^*)\,(K_{-i} - z_i)\,, \quad |\,\psi(0)\rangle = \bigotimes_{i=1}^{k} |\,z_i\rangle \quad .$$



Hypercomputational Quantum Algorithm I

1. Construct a physical process subject to $H_{\mathrm{A}}(t)$ over the time interval $[0, T]$, for some finite time T.
2. Measure through the time-dependent Schrödinger equation $i\partial_t \,|\, \psi(t)\rangle = H_{\mathrm{A}}(t) \,|\, \psi(t)\rangle$, for $t \in [0, T]$ the maximum probability

$$P_{\max}(T) = \max_{(n_1,\ldots,n_k)\in\mathbb{N}^k} |\langle \psi(T) \mid n_1, \ldots, n_k\rangle|^2 = \left|\langle \psi(T) \mid \{n\}\rangle_0\right|^2 .$$

3. If $P_{\max}(T) \leq 1/2$, increase T and repeat all the steps above .
4. If $P_{\max}(T) > 1/2$ (halting criterion) then $|\, \{n\}\rangle_0$ is the ground state of H_{D} (assuming no degeneracy).
5. $D(x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}$, if and only if, $H_{\mathrm{D}} \,|\, \{n\}\rangle_0 = 0$.

Simulation I

Idea

To solve Schrödinger equation in a truncated
$m+1$-dimensional Fock space with computational basis

$$\left\{ |\, n\rangle^{m+1} \mid 0 \leq n \leq m \right\} \quad ,$$

through of unitary Cayley transform on H_A^{m+1}

$$|\, \psi(t+1)\rangle^{m+1} = \frac{1 - \frac{i}{2} H_A^{m+1}(t)}{1 + \frac{i}{2} H_A^{m+1}(t)} \,|\, \psi(t)\rangle^{m+1} \, .$$

Simulation II

Example (Equation with solution)

For the Diophantine equation $D(x) \equiv x - 6 = 0$:

$$|\psi(0)\rangle^{10} \approx [0.16\ 0.36\ 0.51\ 0.53\ 0.43\ 0.29\ 0.17\ 0.08\ 0.04\ 0.02]^T,$$

$$H_{\mathrm{D}}^{10\times 10} = \left(N^{10\times 10} - 6\mathbb{1}^{10\times 10}\right)^2,$$

$$H_{\mathrm{I}}^{10\times 10} = \left(K_+^{10\times 10} - 4\mathbb{1}^{10\times 10}\right)\left(K_-^{10\times 10} - 4\mathbb{1}^{10\times 10}\right).$$

The figure (1) indicates that the maximum probability reaches the value of $1/2$ for $T \approx 35$ and then the $|\{n\}\rangle_0 = |6\rangle^{10} = \begin{bmatrix} 0 & \dots & 1 & 0 & 0 & 0 \end{bmatrix}^T$.

Simulation III

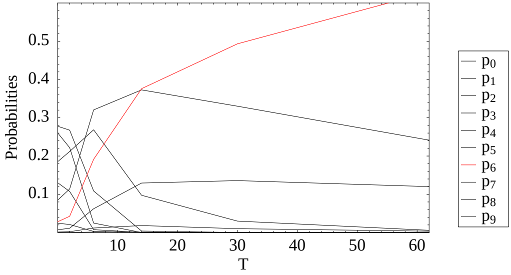


Figure: Simulation results for $D(x) \equiv x - 6 = 0$ with $(z, m) = (4, 9)$.

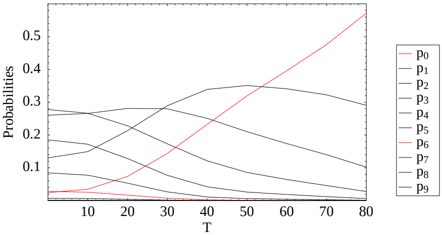
Solution for $D(x) \equiv x - 6 = 0$

$H_{\mathrm{D}}^{10\times10} \mid 6\rangle^{10} = 0$ implies than $D(x)$ has solution in the non-negative integers $\mathbb{N}$.

Simulation IV

Example (Equation without solution $D(x) \equiv x + 6 = 0$)

The figure indicates that $|\{n\}\rangle_0 = |0\rangle^{10}$ for $T \approx 70$ with $(z, m) = (4, 9)$.



Solution for $D(x) \equiv x + 6 = 0$

Due to $H_{\mathrm{D}}^{10\times 10}|0\rangle^{10} \neq 0$, we do not have decision criterion.

Simulation V

Algorithm (physical referent)

5 $D(x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}$, if and only if, $H_\mathrm{D} \,|\, \{n\}\rangle_0 = 0$.

Algorithm (simulation)

5

- **(a)** If $H_\mathrm{D} \,|\, \{n\}\rangle_0 = 0$ then $D(x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}$
- **(b)** If $H_\mathrm{D} \,|\, \{n\}\rangle_0 \neq 0$ then increase m and repeat all the steps above

  $\Longrightarrow$ We do not have halting criterion.

Conclusions

- *"Once upon on time, back in the golden age of the recursive function theory, computability was an absolute"*[1]
- The implementation of a hypercomputer is an open problem.
- Adiabatic continuos quantum computation $\neq$ "standard" quantum computation.
- The numerical simulations cannot be equivalent to physical hypercomputational quantum algorithm, due to the impossibility to simulate an infinite number of dimensions.

[1] Richard Sylvan and Jack Copeland. Computability is logic-relative. In Graham Priest and Dominic Hyde, editors, Sociative logics and their applications: essays by the late Richard Sylvan. London: Ashgate Publishing Company, 2000, p. 189.

Further reading I

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Further reading II

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A possible hypercomputational quantum algorithm.
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Acknowledgments and contact

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