Computability and Parallelism

Andrés Sicard-Ramírez

Ciclo de Conferencias
Centro de Computación Científica Apolo
Universidad EAFIT
2018-09-19

Motivation

Question

Does parallelism increase the set of functions that can be computed?

Abstract/Outline

It is accepted that the λ-calculus is a model of computation. It is also known that Plotkin's `parallel-or` function or Church's δ function are not λ-definable. We discuss if some extensions of the λ-calculus, where these functions are definable, contradict the Church-Turing thesis.

Lambda Calculus

Alonzo Church (1903 – 1995)*

*Figures sources: History of computers, Wikipedia and MacTutor History of Mathematics .

Lambda Calculus

Some remarks

- A formal system invented by Church around 1930s.
- The goal was to use the λ-calculus in the foundation of mathematics.
- Intended for studying functions and recursion.
- Model of computation.
- A free-type functional programming language.
- λ-notation (e.g., anonymous functions and currying).

Lambda Calculus

Informally

λ-calculus	Example	Represent
Variable	x	x
Abstraction	$\lambda x.x^2 + 1$	$f(x) = x^2 + 1$
Application	$(\lambda x.x^2 + 1)3$	$f(3)$
β-reduction	$(\lambda x.x^2 + 1)3 =_\beta x^2 + 1[\,x := 3\,] \equiv 10$	$f(3) = 10$

Definition

The set of **$\boldsymbol{\lambda}$-terms** can be defined by an abstract grammar.

$$t ::= x \mid t\,t \mid \lambda x.t$$

Lambda Calculus

Conventions and syntactic sugar

- The symbol '$\equiv$' denotes the syntactic identity.
- Outermost parentheses are not written.
- Application has higher precedence, i.e.,

$$\lambda x.MN \equiv (\lambda x.(MN)).$$

- Application associates to the left, i.e.,

$$MN_1 \ldots N_k \equiv (\ldots((MN_1)N_1)\ldots N_k).$$

- Abstraction associates to the right, i.e.,

$$\begin{aligned}\lambda x_1 x_2 \ldots x_n.M &\equiv \lambda x_1.\lambda x_2.\ldots.\lambda x_n.M \\ &\equiv (\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.M)\ldots))).\end{aligned}$$

Lambda Calculus

Example

Some λ-terms.

- xx (self-application)
- $\mathsf{I} \equiv \lambda x.x$ (identity operator)
- true $\equiv \lambda xy.x$
- false $\equiv \lambda xy.y$
- zero $\equiv \lambda fx.x$
- succ $\equiv \lambda nfx.f(nfx)$
- $\lambda f.VV$, where $V \equiv \lambda x.f(xx)$ (fixed-point operator)
- $\Omega \equiv ww$, where $\omega \equiv \lambda x.xx$.

Lambda Calculus

Definition

A variable x occurs **free** in M if x is not in the scope of λx. Otherwise, x occurs **bound**.

Notation

The result of substituting N for every free occurrence of x in M, and changing bound variables to avoid clashes, is denoted by $M[\,x := N\,]$.*

*See, e.g., Hindley and Seldin [2008, Definition 1.12].

Lambda Calculus

Definition

A **combinator** (or **closed $\boldsymbol{\lambda}$-term**) is a λ-term without free variables.

Convention

A combinator called for example succ will be denoted by succ.

Remark

The programs in a programming language based on λ-calculus are combinators.

Lambda Calculus

Conversion rules

The functional behaviour of the λ-calculus is formalised through of their conversion rules:

$$\lambda x.N =_\alpha \lambda y.(N[\,x := y\,]) \qquad (\alpha\text{-conversion})$$

$$(\lambda x.M)N =_\beta M[\,x := N\,] \qquad (\beta\text{-conversion})$$

$$\lambda x.Mx =_\eta M \qquad (\eta\text{-conversion})$$

Lambda Calculus

Example

Some examples of β-equality (or β-convertibility).

- $\mathsf{I}\ M =_\beta M$
- $\mathsf{succ\ zero} =_\beta \lambda fx.fx \equiv \mathsf{one}$
- $\mathsf{succ\ one} =_\beta \lambda fx.f(fx) \equiv \mathsf{two}$
- $\Omega \equiv (\lambda x.xx)(\lambda x.xx) =_\beta \Omega =_\beta \Omega =_\beta \Omega \dots$

Lambda Calculus

Definition

A $\boldsymbol{\beta}$**-redex** is a λ-term of the form $(\lambda x.M)N$.

Definition

A λ-term which contains no β-redex is in $\boldsymbol{\beta}$**-normal form** (β-nf).

Definition

A λ-term N **is a** $\boldsymbol{\beta}$**-nf of** M (or M **has the** $\boldsymbol{\beta}$**-nf** M) iff N is a β-nf and $M =_\beta N$.

Lambda Calculus

Theorem

Church [1935, 1936] proved that the set

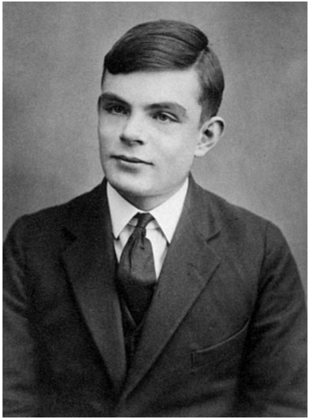
$$\{M \in \lambda\text{-term} \mid M \text{ has a } \beta\text{-normal form}\}$$

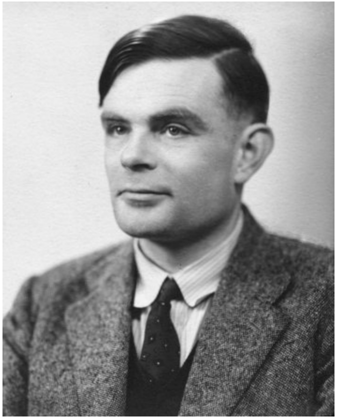
is not computable.* This was the first not computable (undecidable) set ever.†

*We use the term ‘computable‘ rather than ‘recursive‘ following to Soare [1996].
†See also Barendregt [1990].

The Church-Turing Thesis

Alan Mathison Turing (1912 – 1954)*

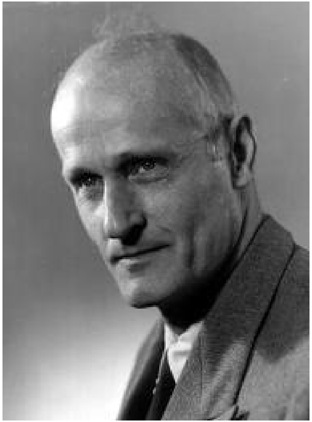


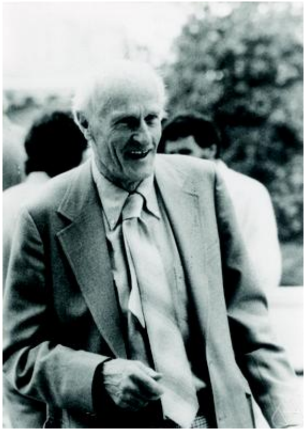


*Figures sources: Wikipedia and National Portrait Gallery .

The Church-Turing Thesis

Stephen Cole Kleene (1909 – 1994)*





*Figures sources: MacTutor History of Mathematics and Oberwolfach.

The Church-Turing Thesis

Theorem

The following sets are coextensive:

i) λ-definable functions,

ii) functions computable by a Turing machine and

iii) general recursive functions.

The Church-Turing Thesis

Common versions of the Church-Turing thesis

"A function is computable (effectively calculable) if and only if there is a Turing machine which computes it." [Galton 2006, p. 94]

"The unprovable assumption that any general way to compute will allow us compute only the partial-recursive functions (or equivalently, what Turing machines or modern-day computers can compute) is know as Church's hypothesis or the Church-Turing thesis." [Hopcroft, Motwani and Ullman 2007, p. 236]

The Church-Turing Thesis

Historical remark

The Church-Turing thesis was not stated by Church nor Turing (they stated definitions) but by Kleene.*

An imprecision

Church [1936] and Turing [1936–1937] definitions were in relation to a computor (human computer).

*See, e.g., Soare [1996] and Copeland [2002].

The Church-Turing Thesis

A better version of the Church-Turing thesis

> *"Any procedure than can be carried out by an idealised human clerk working mechanically with paper and pencil can also be carried out by a Turing machine." [Copeland and Sylvan 1999]*

The Church-Turing Thesis

A better version of the Church-Turing thesis

> *"Any procedure than can be carried out by an idealised human clerk working mechanically with paper and pencil can also be carried out by a Turing machine." [Copeland and Sylvan 1999]*

Question

Why are we talking about "versions" of the Church-Turing thesis?

The Church-Turing Thesis

A better version of the Church-Turing thesis

> *"Any procedure than can be carried out by an idealised human clerk working mechanically with paper and pencil can also be carried out by a Turing machine." [Copeland and Sylvan 1999]*

Question

Why are we talking about "versions" of the Church-Turing thesis?

A/ Because the term 'Church-Turing thesis' was first named, but not defined, by Kleene in 1952 [Jay and Vergara 2004].

Plotkin's `parallel-or` Function

Definition

Let a be an arbitrary type and let f and $\bot$ be a terminating and a non-terminating function from a to a, respectively. Plotkin [1977] **`parallel-or` function** has the following behaviour:

$$\begin{aligned}
&\mathrm{pOr} :: (a \to a) \to (a \to a) \to a \to a \\
&\mathrm{pOr}\ f\ \bot = f \\
&\mathrm{pOr}\ \bot\ f = f \\
&\mathrm{pOr}\ \bot\ \bot = \bot
\end{aligned}$$

*http://hackage.haskell.org/package/unamb .

Plotkin's `parallel-or` Function

Definition

Let a be an arbitrary type and let f and $\perp$ be a terminating and a non-terminating function from a to a, respectively. Plotkin [1977] **`parallel-or` function** has the following behaviour:

$$\begin{aligned}
&\mathrm{pOr} :: (a \to a) \to (a \to a) \to a \to a \\
&\mathrm{pOr}\ f\ \perp = f \\
&\mathrm{pOr}\ \perp\ f = f \\
&\mathrm{pOr}\ \perp\ \perp = \perp
\end{aligned}$$

Haskell implementation

See the `unamb` function from the unambiguous choice library.*

*http://hackage.haskell.org/package/unamb .

Plotkin's `parallel-or` Function

Definition

From Sun's Multithreaded Programming Guide:*

> "***Parallelism:*** *A condition that arises when at least two threads are executing simultaneously.*"
>
> "***Concurrency:*** *A condition that exists when at least two threads are making progress. A more generalized form of parallelism that can include time-slicing as a form of virtual parallelism.*"

---

* https://docs.oracle.com/cd/E19455-01/806-5257/6je9h032b/index.html .

Plotkin's `parallel-or` Function

Definition

From Sun's Multithreaded Programming Guide:*

> "***Parallelism:*** *A condition that arises when at least two threads are executing simultaneously.*"
>
> "***Concurrency:*** *A condition that exists when at least two threads are making progress. A more generalized form of parallelism that can include time-slicing as a form of virtual parallelism.*"

Question

Are we talking about a parallel or concurrent function?

*`https://docs.oracle.com/cd/E19455-01/806-5257/6je9h032b/index.html` .

Plotkin's parallel-or Function

Theorem

The parallel-or function is an effectively calculable function which is not λ-definable [Plotkin 1977].*

---

*See, also, Turner [2006].

Church's δ Function

Definition

Let Δ be the set of λ-terms, let $\equiv$ be the syntactic identity on λ-terms and let M and N be two combinators in β-normal form. **Church's δ function** is defined by

$$\delta M N = \begin{cases} \textsf{true}, & \text{if } M \equiv N; \\ \textsf{true}, & \text{if } M \not\equiv N. \end{cases}$$

Theorem

Church's δ function is not λ-definable [Barendregt 2004, Corollary 20.3.3, p. 520].

Extensions of Lambda Calculus

Jay and Vergara [2017] wrote (emphasis is ours):

> *"For over fifteen years, the lead author has been developing calculi that are more expressive than λ-calculus, beginning with the constructor calculus [8], then pattern calculus [2,7,3], SF-calculus [6] and now λSF-calculus [5]...*
>
> *[The] λSF-calculus is able to query programs expressed as λ-abstractions, as well as combinators, something that is beyond pure λ-calculus.*
>
> *In particular, we have proved (and verified in Coq [4]) that equality of closed normal forms is definable within λSF-calculus."*

Extensions of Lambda Calculus

Jay and Vergara [2017] also stated the following corollaries:

1. Church's δ is λSF-definable.
2. Church's δ is λ-definable.
3. Church's δ is not λ-definable.

Discussion

Question

Do Plotkin's `parallel-or` function or Church's δ function—which are effectively calculable functions but they are not λ-definable functions—contradict the Church-Turing thesis?

Discussion

Question

Do Plotkin's `parallel-or` function or Church's δ function—which are effectively calculable functions but they are not λ-definable functions—contradict the Church-Turing thesis?

A/ No! But we need a better version of the Church-Turing thesis.

Discussion

Definition

A function f is a **number-theoretical function** iff

$$f : \mathbb{N}^k \to \mathbb{N}, \text{ with } k \in \mathbb{N}.$$

Discussion

Definition

A function f is a **number-theoretical function** iff

$$f : \mathbb{N}^k \to \mathbb{N}, \text{ with } k \in \mathbb{N}.$$

Theorem

The following sets are coextensive:

i) λ-definable number-theoretical functions,
ii) number-theoretical functions computable by a Turing machine and
iii) general recursive functions.

Remark

The above theorem is historically precise as pointed out in [Jay and Vergara 2004].

Discussion

A better version of the Church-Turing thesis

We should define the Church-Turing thesis by:

Any number-theoretical function than can be computed by an idealised human clerk working mechanically with paper and pencil can also be computed by a Turing machine.

Discussion

A better version of the Church-Turing thesis

We should define the Church-Turing thesis by:

Any number-theoretical function than can be computed by an idealised human clerk working mechanically with paper and pencil can also be computed by a Turing machine.

Remark

Jay and Vergara [2004, 2017] also negatively answer the question under discussion stating other versions of the Church-Turing thesis.

References

Barendregt, H. P. [1984] (2004). The Lambda Calculus. Its Syntax and Semantics. Revised edition, 6th impression. Vol. 103. Studies in Logic and the Foundations of Mathematics. Elsevier (cit. on p. 28).

Barendregt, Henk (1990). Functional Programming and Lambda Calculus. In: Handbook of Theoretical Computer Science. Ed. by van Leeuwen, J. Vol. B. Formal Models and Semantics. MIT Press. Chap. 7. DOI: 10.1016/B978-0-444-88074-1.50012-3 (cit. on p. 14).

Church, Alonzo (1935). An Unsolvable Problem of Elementary Number Theory. Preliminar Report (Abstract). Bulletin of the American Mathematical Society 41.5, pp. 332–333. DOI: 10.1090/S0002-9904-1935-06102-6 (cit. on p. 14).

— (1936). An Unsolvable Problem of Elementary Number Theory. American Journal of Mathematics 58.2, pp. 345–363. DOI: 10.2307/2371045 (cit. on pp. 14, 19).

Copeland, B. Jack (2002). Hypercomputation. Minds and Machines 12.4, pp. 461–502. DOI: 10.1023/A:1021105915386 (cit. on p. 19).

Copeland, B. Jack and Sylvan, Richard (1999). Beyond the Universal Turing Machine. Australasian Journal of Philosophy 77.1, pp. 44–66. DOI: 10.1080/00048409912348801 (cit. on pp. 20–22).

References

Galton, Antony (2006). The Church-Turing Thesis: Still Valid after All These Years? Applied Mathematics and Computation 178.1, pp. 93–102. DOI: 10.1016/j.amc.2005.09.086 (cit. on p. 18).

Hindley, J. Roger and Seldin, Jonathan P. (2008). Lambda-Calculus and Combinators. An Introduction. Cambridge University Press (cit. on p. 9).

Hopcroft, John E., Motwani, Rajeev and Ullman, Jefferey D. [1979] (2007). Introduction to Automata theory, Languages, and Computation. 3rd ed. Pearson Education (cit. on p. 18).

Jay, Barry and Vergara, Jose (2004). Confusion in the Church-Turing Thesis. Draft version. URL: https://arxiv.org/abs/1410.7103 (cit. on pp. 20–22, 33–36).

— (2017). Conflicting Accounts of λ-Definability. Journal of Logical and Algebraic Methods in Programming 87, pp. 1–3. DOI: 10.1016/j.jlamp.2016.11.001 (cit. on pp. 29, 30, 35, 36).

Plotkin, G. D. (1977). LCF Considered as a Programming Language. Theoretical Computer Science 5.3, pp. 223–255. DOI: 10.1016/0304-3975(77)90044-5 (cit. on pp. 23, 24, 27).

Soare, Robert I. (1996). Computability and Recursion. The Bulletin of Symbolic Logic 2.3, pp. 284–321. DOI: 10.2307/420992 (cit. on pp. 14, 19).

References

Turing, Alan M. (1936–1937). On Computable Numbers, with an Application to the Entscheidungsproblem. Proceeding of the London Mathematical Society s2-42, pp. 230–265. DOI: 10.1112/plms/s2-42.1.230 (cit. on p. 19).

Turner, David (2006). Church's Thesis and Functional Programming. In: Church's Thesis After 70 Years. Ed. by Olszewski, Adam, Woleński, Jan and Janusz, Robert. Ontos Verlag, pp. 518–544. DOI: 10.1515/9783110325461.518 (cit. on p. 27).

Thanks!