Hypercomputation

Andrés Sicard-Ramírez
(asicard@eafit.edu.co)

Grupo de Lógica y Computación
Universidad EAFIT
Medellín, Colombia

Instituto de Computación
Universidad de la República
Montevideo, Uruguay

The Weekly Programming Logic Group Meeting
Chalmers University of Technology, Göteborg, Sweden
30/09/2006

Abstract

While the idea of an absolute computability (i.e. Turing machine computability), detached from logical, mathematical, physical or biological theories is hard to hold nowadays, the idea of a relative computability has progressively gained supporters, as shown by the establishment of an academic community around hypercomputation theory (i.e. computing beyond Turing machine's limit). In this talk we will show some hypercomputation models, the relation between hypercomputation and the Church-Turing thesis, and our results on the possibility of hypercomputation based on quantum computation.

Some geographical facts

Colombia

- Area: 1.14 m. sq km
- Pop: 45 million
- Capital: Bogotá
- Language: Spanish
- Religion: Catholic (95%)

Some geographical facts

Medellín

- Second largest, industrial and commercial city in the country.
- Medellín is also called "Ciudad de la Eterna Primavera" (The City of the Eternal Spring), due to its fantastic weather all year long (between 20^0 C and 30^0 C).

The definition

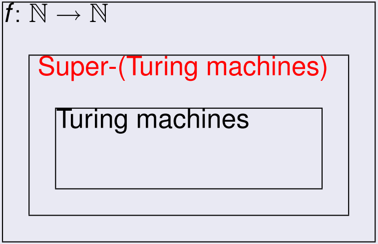
Definition

"A hypercomputer is any machine (theoretical or real) that compute functions or numbers, or more generally solve problems or carry out tasks, that cannot be computed or solved by a Turing machine (TM)." [4].

From the definition: Super-(Turing machines)

Definition

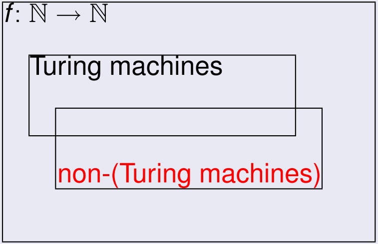
"A hypercomputer that compute anything that a Turing machine can compute, and more, is called a Super-(Turing machine)." [18]



From the definition: non-(Turing-machines)

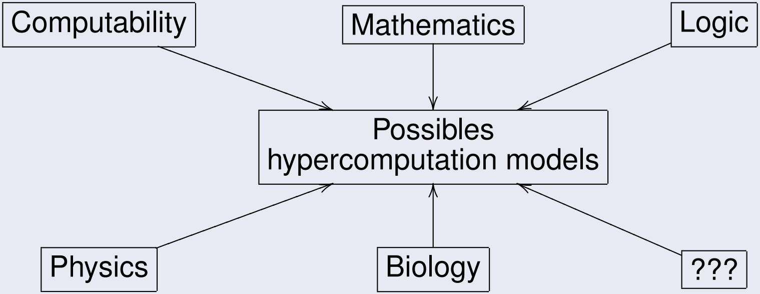
Definition

"A hypercomputer that cannot compute some thing that a universal Turing machine can compute, is called a non-(Turing machine)" [18]



Hypercomputation possibilities

Possible underlying theories



References: [4, 19]

Hypercomputation model: Oracle Turing machines

Definition

A oracle Turing machine (OTM) is a Turing machine equipped with an oracle that is capable of answering questions about the membership of a specific set of natural numbers [22].

Hypercomputational characteristic

- If oracle $\equiv$ recursive set then OTM $\equiv$ TM
- If oracle $\equiv$ non-recursive set then OTM is a hypercomputation model

The term ‘Hypercomputation’

It is precisely due to the existence of Turing’s oracle machines that Jack Copeland introduced the term ‘hypercomputation’ in 1999 [5] to replace wrong expressions such as ‘super-Turing computation’, ‘computing beyond Turing’s limit’, ‘breaking the Turing barrier’, and similar.

Hypercomputation model: Accelerated Turing machines

Definition

An accelerated Turing machine is a Turing machine that performs its first step in one unit of time and each subsequent step in half the time of the step before [3].

Hypercomputational characteristic

Since

$$1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{i=0}^{\infty} \frac{1}{2^n} = 2,$$

the accelerated Turing machine could complete an infinity of steps in two time units.

Hypercomputation model: Analog recurrent neural network (ARNN)

Hypercomputational characteristic

$$\begin{aligned} \text{ARNN with integer weights} &\equiv \text{finite state automata} \\ \text{ARNN with rational weights} &\equiv \text{Turing machines} \\ \text{ARNN with real weights} &\equiv \text{hypercomputation model} \end{aligned}$$

References: [17]

Academic community

- Network:
  Hypercomputation Research Network
  `www.hypercomputation.net`
- Next workshop:
  Future Trends in Hypercomputation
  Sheffield UK, 11-13 September 2006
  `www.hypercomputation.net/hypertrends06/`
- Published special issues:
  - Vols. 12(4) and 13(1) of *Mind and Machines*, 2002 and 2003.
  - Vol. 317(1-3) of *Theoretical Computer Science*, 2004.
  - Vol. 178(1) of *Applied Mathematics and Computation*, 2006

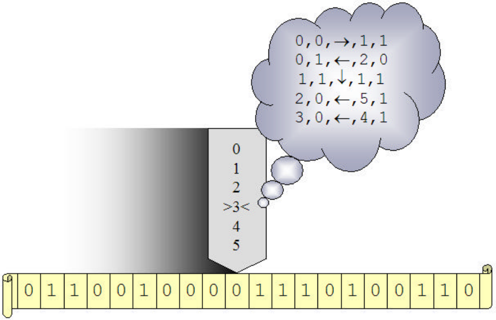
Church's definition

"*We now define the notion, already discussed, of an effectively calculable function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers).*" [2, p. 356]

The Church-Turing thesis and hypercomputation

Turing's definition

"*The "computable" numbers include all numbers which would naturally be regarded as computable.*" [21, p. 249]



Kleene's terminology evolution

(1943): Church's definition $\equiv$ Thesis I [11]
(1952): Church's thesis and Turing's thesis [12]
(1967): Church-Turing thesis [13]

The Church-Turing thesis and hypercomputation

The Church-Turing thesis

"*Any procedure than can be carried out by an idealised human clerk working mechanically with paper and pencil can also be carried out by a Turing machine.*"

The relation

- The current hypercomputation models do not refute the Church-Turing thesis.
- The hypercomputation and the Church-Turing thesis can live together.

The Church-Turing thesis and hypercomputation

The so-called Church-Turing thesis

"*Anything computable is Turing machine computable*".

The relation

- Anything computable $\equiv$ Anything theoretical computable
  The hypercomputation refutes the so-called Church-Turing
- Anything computable $\equiv$ Anything physically computable
  This is a open problem, on fact, this is the problem.

Open problem

The most important open problem

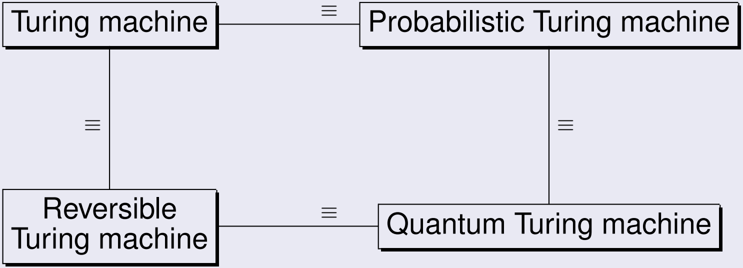
It is possible to build a hypermachine?

- Based on relativistic physics (cosmological singularities) ?
- Based on quantum physics (quantum adiabatic theorem) ?

Standard quantum computation (SQC)

Models

Quantum Turing machines [6] and quantum circuits [7].



"Weak" hypercomputation based on SQC

Generation of truly random numbers

$$U_H \,|\, 0\rangle = \frac{1}{\sqrt{2}}(|\, 0\rangle + |\, 1\rangle) \to \text{measure}$$

1. We observe the superposition state. *"The act of observation causes the superposition to collapse into either $|\, 0\rangle$ or the $|\, 1\rangle$ state with equal probability. Hence you can exploit quantum mechanical superposition and indeterminism to simulate a perfectly fair coin toss."* [23, p. 160]
2. The problem: It is not clear how to use this property to solve to a TM incomputable problem [14].

Others quantum computation models

Common misunderstanding

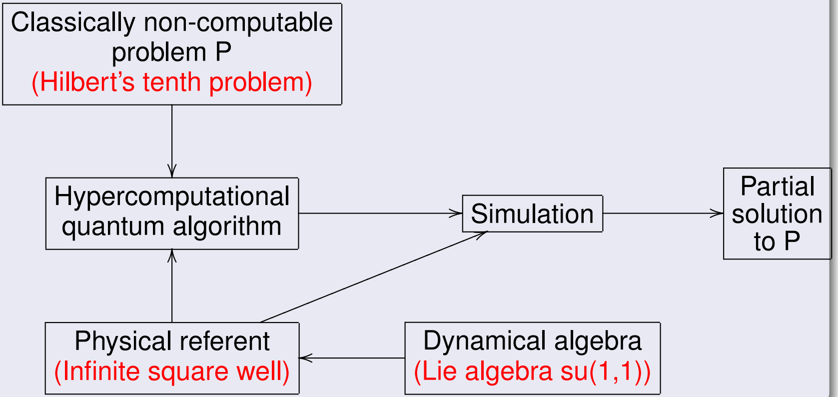
$$\begin{aligned}\text{quantum computation} &\equiv \text{SQC}\\ &\equiv \text{adiabatic quantum computation (AQC)}\end{aligned}$$

The real situation

$$\begin{aligned}\text{finite AQC} &\equiv \text{SQC}\\ \text{infinite AQC} &\equiv \text{Kieu's hypercomputational quantum algorithm}\\ &\qquad \text{(hypercomputation model)}[8, 10, 9]\end{aligned}$$

Hypercomputational quantum algorithm à la Kieu

Key ideas



Incomputable TM problem: Hilbert's tenth problem

Given a Diophantine equation

$$D(x_1, \ldots, x_k) = 0 \ ,$$

we should build a procedure to determine whether or not this equation has a solution in $\mathbb{N}$.

Classical solution (Matiyasevich, Davis, Robinson, Putnam)

$$\boxed{\text{Hilbert's tenth problem}} \equiv \boxed{\text{Halting problem for Turing Machine}}$$

Dynamical algebra $\mathfrak{su}(1,1)$

- Computational basis

$$\mathfrak{F}^{\mathfrak{su}(1,1)} = \{|\, n\rangle \mid n \in \mathbb{N}\}$$

$$|\,0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad |\,1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad |\,2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \qquad \ldots,$$

- Commutation relations

$$[K_-, K_+] = K_3 \quad , \qquad [K_\pm, K_3] = \mp 2K_\pm$$

Hypercomputational quantum algorithm à la Kieu

Dynamical algebra $\mathfrak{su}(1,1)$

- Infinite-dimensional irreducible representation for $\mathfrak{su}(1,1)$

$$K_+ \,|\, n\rangle = \sqrt{(n+1)(n+3)}\,|\, n+1\rangle \text{ (creation operator)} \;,$$
$$K_- \,|\, n\rangle = \sqrt{n(n+2)}\,|\, n-1\rangle \text{ (annihilation operator)} \;,$$
$$K_3 \,|\, n\rangle = (2n+3)\,|\, n\rangle \text{ (Cartan operator)} \;.$$

- Number operator

$$N = (1/2)(K_3 - 3) = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \;,$$

$$N \,|\, n\rangle = n \,|\, n\rangle \;.$$

Hypercomputational quantum algorithm à la Kieu

Codification-decodification

$$
\begin{tikzcd}
 & \boxed{D(x_1, \ldots, x_k) = 0} \arrow[dl, "\text{codification}"'] \arrow[dr, "\text{codification}"] \arrow[dd, leftrightarrow] & \\
\boxed{\begin{array}{c}\text{Solution in } \mathbb{N}^k \\ \equiv \\ H_D \,|\, n\rangle_0 = 0\end{array}} & & \boxed{\begin{array}{c}\text{No solution in } \mathbb{N}^k \\ \equiv \\ H_D \,|\, n\rangle_0 \neq 0\end{array}} \\
 & \boxed{H_D = \left(D\left(N_1, \ldots, N_k\right)\right)^2} \arrow[ul, "\text{decodification}"] \arrow[ur, "\text{decodification}"'] &
\end{tikzcd}
$$

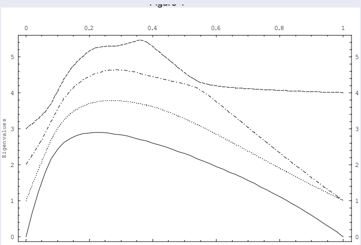
$|\, n\rangle_0$: Ground state of H_D

New problem

To find the ground state $|\,\{n\}\rangle_0$ of H_{D}.

Solution: infinite adiabatic quantum computation

$H_A(t) = (1 - t/T)H_I + (t/T)H_D$, over $t \in [0, T]$



References: [15, 16]

Final remarks

- "*Once upon on time, back in the golden age of the recursive function theory, computability was an absolute.*" [20, p. 189]
- "*Is there any limit to discrete computation, and more generally, to scientific knowledge ?*" [1, p. 1]

Acknowledgments

- We thank Jack Copeland and Tien D. Kieu for helpful discussions and feedback.
- We thank Peter Dybjer (Chalmers university), Ana Bove (Chalmers university), Alberto Pardo (Universidad de la República), and Francisco Correa (Universidad EAFIT) for help me in all issues related to my visit to Chalmers university.
- We thank Universidad EAFIT and LerNET project for financial support.

References I

[1] C. S. Calude and M. J. Dinneen.
Breaking the turing barrier.
Technical report, CDMTCS, 1998.

[2] A. Church.
An unsolvable problem of elementary number theory.
American Journal of Mathematics, 58(2):345–363, 1936.

[3] B. J. Copeland.
Super Turing-machines.
Complexity, 4(1):30–32, 1998.

[4] B. J. Copeland.
Hypercomputation.
Minds and Machines, 12(4):461–502, 2002.

References II

[5] B. J. Copeland and D. Proudfoot.
Alan Turing's forgotten ideas in computer science.
Scientific American, 280(4):76–81, april 1999.

[6] D. Deutsch.
Quantum theory, the Church-Turing principle and the universal quantum computer.
Proc. R. Soc. Lond. A, 400:97–117, 1985.

[7] D. Deutsch.
Quantum computational networks.
Proc. R. Soc. Lond. A, 425:73–90, 1989.

[8] T. D. Kieu.
Computing the non-computable.
Contemporary Physics, 44(1):51–71, 2003.

References III

[9] T. D. Kieu.
Hypercomputation with quantum adiabatic processes.
Theoretical Computer Science, 317(1–3):93–104, 2004.

[10] T. D. Kieu.
A reformulation of Hilbert's tenth problem through quantum mechanics.
Proc. R. Soc. Lond. A, 460:1535–1545, 2004.

[11] S. C. Kleene.
Recursive predicates and quantifiers.
Trans. Amer. Math. Soc., 53:41–73, 1943.

[12] S. C. Kleene.
Introduction to metamathematics.
Amsterdan: North-Holland, 1952.

References IV

[13] S. C. Kleene.
Mathematical Logic.
New York: John Wiley & Sons, 1967.

[14] T. Ord and T. D. Kieu.
Using biased coins as oracles.
Eprint: `arxiv.org/abs/cs/0401019`, 2004.

[15] A. Sicard, J. Ospina, and M. Vélez.
Quantum hypercomputation based on the dynamical algebra $\mathfrak{su}(1,1)$.
J. Phys. A: Math. Gen., 39(40):12539–12558, 2006.

References V

[16] A. Sicard, M. Vélez, and J. Ospina.
A possible hypercomputational quantum algorithm.
In E. J. Donkor, A. R. Pirich, and H. E. Brandt, editors, *Quantum Information and Computation III*, volume 5815 of *Proc. of SPIE*, pages 219–226. SPIE, Bellingham, WA, 2005.

[17] H. T. Siegelmann.
Neural networks and analog computation. Beyond the Turing limit.
Progress in Theorical Computer Science. Boston: Birkhäuser, 1999.

[18] M. Stannett.
Computation and hypercomputation.
Minds and Machines, 13(1):115–153, 2003.

References VI

[19] M. Stannett.
Hypercomputation models.
In C. Teuscher, editor, *Alan Turing: life and legaly of a great thinker*, pages 135–157. Berlin: Springer, 2003.

[20] R. Sylvan and J. Copeland.
Computability is logic-relative.
In G. Priest and D. Hyde, editors, *Sociative logics and their applications: essays by the late Richard Sylvan*, pages 189–199. London: Ashgate Publishing Company, 2000.

[21] A. M. Turing.
On computable numbers, with an application to the Entscheidungsproblem.
Proc. London Math. Soc., 42:230–265, 1936-7.
A correction, ibid, vol. 43, no. 2198, p. 544–546, 1937.

References VII

[22] A. M. Turing.
Systems of logic based on ordinales.
Proc. London Math. Soc., 45(2239):161–228, 1939.

[23] C. P. Williams and S. H. Clearwater.
Explorations in quantum computing.
New York: Springer-Telos, 1997.