Computer Assisted Proofs

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Introduction

A big spectrum

pencil and paper proofs

⋮

formally verified proofs

own computer programs

⋮

automatic theorems provers

⋮

interactive theorems provers

Some families in the spectrum

- Automatic theorems provers for first-order logic
- Satisfiability modulo theories solvers
- Inductive theorem provers
- Interactive proof assistants
- Tools for the verification of programs

Automatic Theorems Provers for First-Order Logic

Some features

- The TPTP World

  'A well know and established infrastructure that supports research, development, and deployment of automated theorem proving systems for classical logics.' [Sutcliffe 2010, p. 1.]

  See http://www.tptp.org/.

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  The World Championship for Automated Theorem Proving!

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- Some ATPs for (classic) first-order logic: E, Equinox, Metis, SPASS and Vampire.

The TPTP Languages

- FOF (Full First-Order Form): Full first-order logic [Sutcliffe 2009].
- CNF (Clause Normal Form): First-order logic in conjunctive clausal form [Sutcliffe 2009].
- TFF (Typed First-Order Form): TF0 (monomorphic) [Sutcliffe, Schulz, Claessen and Baumgartner 2012] and TF1 (rank-1 polymorphism) [Blanchette and Paskevich 2013].
- THF (Typed Higher-Order Form): TH0 (monomorphic) [Sutcliffe and Benzmüller 2010] and TH1 (rank-1 polymorphism) TODO.
- TXF (Typed Extended First-Order Form): TX0 (monomorphic) and TX1 (rank-1 polymorphism) TODO.

FOF Annotated Formulae and Problems

`fof(name, role, formula)` where:

`name` identifies the formula,

`formula` is a FOL-formula and

`role` $\in \{$ `axiom`, `definition`, `hypothesis`, `conjecture` $\}$.

Each TPTP problem contains one or more annotated formulae.

FOF Syntax

Logical constant	Symbol
True	`$true`
False	`$false`
Conjunction	`&`
Disjunction	`|`
Negation	`~`
Conditional	`=>`
Biconditional	`<=>`
Universal quantifier	`![X] : p(X)`
Existential quantifier	`?[X] : p(X)`

Satisfiability Modulo Theories Solvers

Satisfiability Modulo Theories

- The study of automatic methods for checking the satisfiability of first-order formulae with respect to some background theory is called Satisfiability Modulo Theories (SMT) [Barret, Sebastiani, Seshia and Tinelli 2009].

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  Theories of real numbers, integers, arrays, lists, records, bit-vectors,...

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SMT-LIB

The SMT library and standard.

Satisfiability Modulo Theories Solvers (SMT Solvers)

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Z3 demo

See Z3 home page.

Inductive Theorem Provers

Inductive Theorem Provers (ITPs)

- There is not standard

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- There is not standard
- ITPs (Google for List of All Inductive Theorem Provers in The World).

Interactive Proof Assistants

Description

Proof assistants are computer systems that allow a user to do mathematics on a computer, but not so much the computing (numerical or symbolical) aspect of mathematics but the aspects of proving and defining. So a user can set up a mathematical theory, define properties and do logical reasoning with them. [Geuvers 2009, p. 3.]

Some Proof Assistants

- Agda
- Coq
- Isabelle
- Lean

Agda

- Chalmers University of Technology and University of Gothenburg (Sweden)
- Based on Martin-Löf Type Theory
- Direct manipulation of proofs-objects
- Back-ends to Haskell (GHC) and JavaScript
- Written in Haskell

Coq

- INRIA (France)
- Based on the Calculus of Inductive Constructions
- Tactic-based
- Extraction of programs to Haskell, OCaml and Scheme
- Written in OCaml (with a bit of C)

Isabelle

- University of Cambridge (England) and Technical University of Munich (German)
- Based on HOL (Higher Order Logic)
- Tactic-based
- Extraction of programs to Haskell, OCaml, Scala and SML
- Written in SML
- Integration with ATPs and SMT solvers

Lean

- Microsoft Research and Carnegie Mellon University (USA)
- Based on dependent type theory
- Declarative style and tactic-bases
- Written in C++
- Integration with ATPs and SMT solvers

Real Applications

Verification of Landau's *Grundlagen der Analysis*

1977 PhD thesis [van Benthem Jutting 1977] (Automath).

Mizar Mathematical Library (MML)

One of the largest repository of formalised mathematics (1990–current).

MML includes [Bancerek, Byliński, Grabowski, Korniłowicz, Matuszewski, Naumowicz and Pak 2018]:

- over 12000 definitions
- over 59000 theorems
- over 1290 articles
- over 253 authors

Proof of the Four Colour Theorem

1976 Every planar graph is four colorable. Announce in the Bulletin of AMS [K. Appel and Haken 1976].

1977 Description of the proof in Scientific American [K. Appel and Haken 1977].

1977 Simpler proof (still using a computer) [Robertson, Sanders, Seymour and Thomas 1997].

2005 Formalised proof. Microsoft report [Gonthier 2005] (Coq).

2008 Description of the formalised proof in Notices of AMS [Gonthier 2008].

CompCert: A Verified Compiler for a Large Subset of C

2006 Journal (and proceedings) publication: POPL 2006 (Coq).

2009 Description in CACM [Leroy 2009].

seL4: Formal Verification of an Operating System Kernel

2009 Proceedings of SOSP 2009 (Isabelle/HOL).

2010 Description in CACM [Klein, Andronick, Elphinstone, Heiser, Cock, Derrin, Elkaduwe, Engelhardt, Kolanski, Norrish, T. Sewell, Tuch and Winwood 2010].

Proof of the Odd-Order Theorem

1963 Solvability of groups of odd-order [Feit and Thompson 1963].

2012 Announce of the formalised proof (Gonthier and Théry 2012) (Coq).

2013 Formalised proof. Journal publication [Gonthier, Asperti, Avigad, Bertot, Cohen, Garillot, Le Roux, Mahboubi, O'Connor, Biha, Pasca, Rideau, Solovyev, Tassi and Théry 2013].

Proof of Kepler Conjecture

1998 Hales announced proof of Kepler Conjecture.

2005 Journal publication 'uncertified' (Hales 2005).

2014 Announce of the formalised proof (Hales 2014) (Isabelle and Isabelle/HOL, HOL Light, OCaml, CamlP5 and GLPK).

2017 Formalised proof. Journal publication [Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, Magron, McLaughlin, T. T. Nguyen, Q. T. Nguyen, Nipkow, Steven, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu and Zumkeller 2017].

2014 Journal publication [Marché 2014] (Frama-C).

2014 Journal publication [Kumar, Myreen, Norrish and Owens 2014] (HOL4).

Verification of a Cryptographic Primitive: SHA-256

2015 Journal publication [A. W. Appel 2015] (Coq).

Certifying a File System using Crash Hoare Logic

2015 Proceedings of SOSP 2015 (Coq).

2017 Description in CACM [Chajed, Chen, Chlipala, Kaashoek, Zeldovich and Ziegler 2017].

Rigorous Test-Oracle Specification and Validation for TCP/IP and the Sockets API

2001 Technical report (HOL4).

2015 Proceedings publication (2008, 2006, 2005, 2002, 2001).

2017 Journal publication [Bishop, Fairbairn, Mehnert, Norrish, Ridge, P. Sewell, Smith and Wansbrough 2018].

2019 Proceedings of PLDI 2019 [Lööw, Kumar, Tan, Myreen, Norrish, Abrahamsson and Fox 2019] (HOL4).

Proof-Checking Euclid Book I

2019 Journal publication [Beeson, Narboux and Wiedijk 2019] (HOL Light and Coq).

2022 Description in The American Mathematical Monthly [Beeson 2022].

Resolution of Keller's Conjecture

2020 Proceedings of IJCAR 2020 (SMT solver).

2021 Description in CACM [Monroe 2021].

2020 Proceedings of IJCAR 2020 (Isabelle/HOL).

2022 Journal publication [Eßmann, Nipkow, Robillard and Sulejmani 2022].

Tools for the Verification of Programs

Dedicate Tools and Systems for the Verification of Functional Programs

- Hip: The Haskell Inductive Prover

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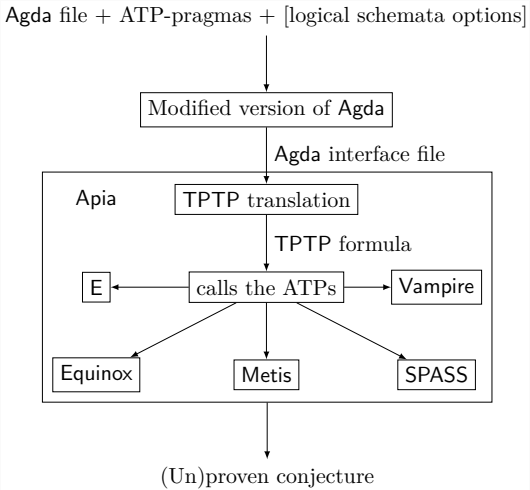
- Sparkle: An interactive proof assistant for Clean

Dedicate Tools and Systems for the Verification of Functional Programs

- Hip: The Haskell Inductive Prover
- Sparkle: An interactive proof assistant for Clean
- Zeno: An automated proof system for Haskell program properties

Combining Agda with ATPs*



References

References

- Appel, A. W. (2015). **'Verification of a Cryptographic Primitive: SHA-256'.** In: *ACM Transactions on Programming Languages and Systems* 37.2, 7:1–31. DOI: 10.1145/2701415 (cit. on p. 39).
- Appel, K. and W. Haken (1976). **'Every Planar Graph is Four Colorable'.** In: *Bulletin of the American Mathematical Society* 8.5, pp. 711–712. DOI: 10.1090/bull/1556 (cit. on p. 32).
- Appel, K. and W. Haken (Oct. 1977). **'The Solution of the Four-Color-Map Problem'.** In: *Scientific American*, pp. 108–121. DOI: 10.1038/scientificamerican1077-108 (cit. on p. 32).

References

- Bancerek, G., C. Byliński, A. Grabowski, A. Korniłowicz, R. Matuszewski, A. Naumowicz and K. Pak (2018). **'The Role of the Mizar Mathematical Library for Interactive Proof Development in Mizar'.** In: *Journal of Automated Reasoning* 61.1–4, pp. 9–32. DOI: 10.1007/s10817-017-9440-6 (cit. on p. 31).
- Barret, C., R. Sebastiani, S. A. Seshia and C. Tinelli (2009). **'Satisfiability Module Theories'.** In: *Handbook of Satisfiability*. Ed. by Biere, A., Heule, M., van Maaren, H. and Walsh, T. IOS Press. Chap. 26 (cit. on pp. 13–15).
- Beeson, M. (2022). **'Euclid after Computer Proof-Checking'.** In: *The American Mathematical Monthly* 129.7, pp. 623–646. DOI: 10.1080/00029890.2022.2069985 (cit. on p. 43).
- Beeson, M., J. Narboux and F. Wiedijk (2019). **'Proof-Checking Euclid'.** In: *Annals of Mathematics and Artificial Intelligence* 85.2–4, pp. 213–257. DOI: 10.1007/s10472-018-9606-x (cit. on p. 43).

References

- Bishop, S., M. Fairbairn, H. Mehnert, M. Norrish, T. Ridge, P. Sewell, M. Smith and K. Wansbrough (2018). **'Engineering with Logic: Rigorous Test-Oracle Specification and Validation for TCP/IP and the Sockets API'.** In: *Journal of the ACM* 66.1, pp. 1–77. DOI: 10.1145/3243650 (cit. on p. 41).
- Blanchette, J. C. and A. Paskevich (2013). **'TFF1: The TPTP Typed First-Order Form with Rank-1 Polymorphism'.** In: *Automated Deduction (CADE-24)*. Ed. by Bonacina, M. P. Vol. 7898. Lecture Notes in Artifical Intellingence. Springer, pp. 414–420. DOI: 10.1007/978-3-642-38574-2_29 (cit. on p. 9).
- Chajed, T., H. Chen, A. Chlipala, M. F. Kaashoek, N. Zeldovich and D. Ziegler (2017). **'Certifying a File System Using Crash Hoare Logic: Correctness in the Presence of Crashes'.** In: *Communications of the ACM* 60.4, pp. 75–84. DOI: 10.1145/3051092 (cit. on p. 40).

References

- Eßmann, R., T. Nipkow, S. Robillard and U. Sulejmani (2022). **'Verified Approximation Algorithms'.** In: *Logical Methods in Computer Science* 18.1, 36:1–36:21. DOI: 10.46298/lmcs-18(1:36)2022 (cit. on p. 45).
- Feit, W. and J. G. Thompson (1963). **'Solvability of Groups of Odd Order'.** In: *Pacific Journal of Mathematics* 13.3 (cit. on p. 35).
- Geuvers, H. (2009). **'Proof Assistants: History, Ideas and Future'.** In: *Sadhana* 34.1, pp. 3–25. DOI: 10.1007/s12046-009-0001-5 (cit. on p. 23).
- Gonthier, G. (2005). ***A Computer-Checked Proof of The Four Colour Theorem*.** Tech. rep. Microsoft report (cit. on p. 32).
- — (2008). **'Formal Proof—The Four Colour Theorem'.** In: *Notices of the AMS* 55.11, pp. 1382–1393 (cit. on p. 32).

References

- Gonthier, G., A. Asperti, J. Avigad, Y. Bertot, C. Cohen, F. Garillot, S. Le Roux, A. Mahboubi, R. O'Connor, S. O. Biha, I. Pasca, L. Rideau, A. Solovyev, E. Tassi and L. Théry (2013). **'A Machine-Checked Proof of the Odd Order Theorem'.** In: *4th International Conference on Interactive Theorem Proving (ITP 2013)*. Ed. by Blazy, S., Paulin-Mohring, C. and Pichardie, D. Vol. 7998. Lecture Notes in Computer Science. Springer, pp. 163–179. DOI: 10.1007/978-3-642-39634-2_14 (cit. on p. 35).
- Hales, T., M. Adams, G. Bauer, T. D. Dang, J. Harrison, L. T. Hoang, C. Kaliszyk, V. Magron, S. McLaughlin, T. T. Nguyen, Q. T. Nguyen, T. Nipkow, O. Steven, J. Pleso, J. Rute, A. Solovyev, T. H. A. Ta, N. T. Tran, T. D. Trieu, J. Urban, K. Vu and R. Zumkeller (2017). **'A Formal Proof of the Kepler Conjecture'.** In: *Forum of Mathematics, Pi* 5, e2. DOI: 10.1017/fmp.2017.1 (cit. on p. 36).

References

- Klein, G., J. Andronick, K. Elphinstone, G. Heiser, D. Cock, P. Derrin, D. Elkaduwe, K. Engelhardt, R. Kolanski, M. Norrish, T. Sewell, H. Tuch and S. Winwood (2010). **'seL4: Formal Verification of an Operating-system Kernel'.** In: *Communications of ACM* 53.6, pp. 107–115. DOI: 10.1145/1743546.1743574 (cit. on p. 34).
- Kumar, R., M. O. Myreen, M. Norrish and S. Owens (2014). **'CakeML: A Verified Implementation of ML'.** In: *Proceedings of the 41th annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2014)*, pp. 179–191. DOI: 10.1145/2535838.2535841 (cit. on p. 38).
- Leroy, X. (2009). **'Formal Verification of a Realistic Compiler'.** In: *Communications of the ACM* 52.7, pp. 107–115. DOI: 10.1145/1538788.1538814 (cit. on p. 33).

References

- Lööw, A., R. Kumar, Y. K. Tan, M. O. Myreen, M. Norrish, O. r. Abrahamsson and A. Fox (2019). **'Verified Compilation on a Verified Processor'.** In: *Proceedings of the 40th ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI 2019)*. ACM, pp. 1041–1053. DOI: 10.1145/3314221.3314622 (cit. on p. 42).
- Marché, C. (2014). **'Verification of the functional behavior of a floating-point program: An industrial case study'.** In: *Science of Computer Programming* 96, Part 3, pp. 279–296. DOI: 10.1016/j.scico.2014.04.003 (cit. on p. 37).
- Monroe, D. (2021). **'A Satisfying Result'.** In: *Communications of the ACM* 64.5, pp. 10–12. DOI: 10.1145/3453650 (cit. on p. 44).
- Robertson, N., D. Sanders, P. Seymour and R. Thomas (1997). **'The Four-Colour Theorem'.** In: *Journal of Combinatorial Theory, Series B* 70.1, pp. 2–44. DOI: 10.1006/jctb.1997.1750 (cit. on p. 32).

References

- Sicard-Ramírez, A. (2015). **'Reasoning about Functional Programs by Combining Interactive and Automatic Proofs'.** PhD thesis. PEDECIBA Informática. Universidad de la República. Uruguay (cit. on p. 50).
- Sutcliffe, G. (2009). **'The TPTP Problem Library and Associated Infrastructure. The FOT and CNF Parts, v3.5.0'.** In: *Journal of Automated Reasoning* 43.4, pp. 337–362. DOI: 10.1007/s10817-017-9407-7 (cit. on p. 9).
- — (2010). **'The TPTP World—Infrastructure for Automated Reasoning'.** In: *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR-16)*. Ed. by Clarke, E. and Voronkov, A. Vol. 6355. Lecture Notes in Computer Science. Springer, pp. 1–12. DOI: 10.1007/978-3-642-17511-4_1 (cit. on pp. 6–8).
- Sutcliffe, G. and C. Benzmüller (2010). **'Automated Reasoning in Higher-Order Logic using the TPTP THF Infrastructure'.** In: *Journal of Formalized Reasoning* 3.1, pp. 1–27. DOI: 10.6092/issn.1972-5787/1710 (cit. on p. 9).

References

- Sutcliffe, G., S. Schulz, K. Claessen and P. Baumgartner (2012). **'The TPTP Typed First-Order Form with Arithmetic'.** In: *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR-18)*. Ed. by Bjørner, N. and Voronkov, A. Vol. 7180. Lecture Notes in Computer Science. Springer, pp. 406–419. DOI: 10.1007/978-3-642-28717-6_32 (cit. on p. 9).
- van Benthem Jutting, L. S. (1977). **'Checking Landau's "Grundlagen" in the Automath System'.** PhD thesis. Technische Hogeschool Eindhoven. DOI: 10.6100/IR23183 (cit. on p. 30).