QUANTUM ALGORITHM OF HYPERCOMPUTATION BASED ON THE PÖSCHL-TELLER POTENTIAL

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RESUMEN

Se construye un algoritmo cuántico de hipercomputación con base en los potenciales Pöschl-Teller, el álgebra de Lie su(1,1) y una evolución adiabática. Este algoritmo resuelve en principio el décimo problema de Hilbert, un problema clásicamente no computable. Este algoritmo es una generalización del algoritmo propuesto por Tien D. Kieu, con base en el oscilador armónico; y del algoritmo propuesto por los autores, con base en la caja de potencial infinita.

Palabras claves: Hipercomputación, computación cuántica, décimo problema de Hilbert,

ABSTRACT

We constructed an hypercomputational quantum algorithm based on Pöschl-Teller potentials, Lie algebra su(1,1), and an adiabatic evolution. Our algorithm resolves in principle Hilbert's tenth problem, a classically non-computable problem. Our algorithm is an adaptation of Tien D. Kieu' s algorithm, which is base on quantum harmonic oscillator; and it is an generalization of our previous algorithm based on the infinite square well.

Keywords: Hypercomputation, quantum computation, Hilbert's tenth problem.

Introduction

The purpose of this work is to present an algorithm of quantum hypercomputation *à la* Kieu [2] for Hilbert's tenth problem, which contributes important considerations on the role that ∞-dimensional unitary irreducible representation (UIR) of dynamical algebras play in the hypercomputational context. The way to proceed is to substitute Weyl-Heisenberg algebra realized on quantum harmonic oscillator for non-compact Lie algebra $\mathfrak{su}(1,1)$ realized on Pöschl-Teller potentials (PT) [1]. Furthermore we presented as a case limit of our algorithm on PT potentials, our hypercomputational algorithm on infinite square well presented previously [3, 4].

Hilbert's Tenth Problem and Kieu's Algorithm

Kieu's idea is essentially to transform Hilbert's tenth problem in the realm of the theory of numbers, into a quantum problem in the realm of the spectral theory and to resolve this problem using the adiabatic theorem. Kieu's algorithm incorporates the following elements: (i) A physical quantum referent, (ii) An algebraic structure realized on the physical quantum referent, a dynamical algebra, (iii) A codification scheme of the Diophantine equation, (iv) An initialization system of the quantum system, (v) A quantum adiabatic evolution process, (vi) A measuring procedure of observable quantum, (vii) A halting criterion and (viii) A decodification scheme to determine the solution to Hilbert tenth problem.

PT Potentials

Consider the continuously indexed family of potentials [1]

$$V_{\lambda,\kappa}(x) = \frac{V_0}{2}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2l} + \frac{\kappa(\kappa-1)}{\sin^2 x/2l}\right), \tag{1}$$

where $0 \leq x \leq \pi l$, the continuous parameters $\lambda, \kappa > 1$ and the coupling constant $V_0 = \hbar^2/4ml^2$. The Hamiltonian

$$H = i^2 \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{\lambda,\kappa}(x) - \frac{V_0}{2}(\lambda+\kappa)^2, \tag{2}$$

corresponds to a particle of mass m subject to the interaction of the PT potentials. The wave function $\Psi_n(x)$ in the representation of coordinates is defined

$$\Psi_n(x) \equiv \langle x \,|\, \eta/2, n\rangle\,, \quad 0 \leq x \leq \pi l, \tag{3}$$

where $\eta = \lambda + \kappa + 1$ and Hamiltonian's action over its normalized eigenvalues is

$$H \,|\, \eta/2, n\rangle = E_n \,|\, \eta/2, n\rangle\,. \tag{4}$$

The spectrum of values of the energy associated to (2) crucially depends on parameters λ, κ

$$E_n = \hbar\omega e_n(\lambda,\kappa), \tag{5}$$

where $\omega = \hbar/2ml^2$, and $e_n(\lambda,\kappa) = n(n+\lambda+\kappa)$. The generators of the dynamical algebra are constructed based on the PT potentials, having as a starting point the spectral structure defined in (5) and according to the following criteria [1]

$$K_+ \,|\, \alpha, n\rangle = \sqrt{e_{n+1}(\lambda,\kappa)}, K_- \,|\, \alpha, n\rangle = \sqrt{e_n(\lambda,\kappa)}\,|\, \alpha, n-1\rangle\,, K_3 \,|\, \alpha, n\rangle = e'(\lambda,\kappa)\,|\, \alpha, n\rangle\,, \tag{6}$$

where $\alpha = \eta/2, e'(\lambda,\kappa) = e_{n+1}(\lambda,\kappa) - e_n(\lambda,\kappa)$, the operators K_+, K_- y K_3 are called creation, annihilation and Cartan operators respectively. Those operators satisfy the commutation relations of Lie algebra $\mathfrak{su}(1,1)$ given by [1]

$$[K_\pm, K_3] = \mp 2K_\pm,\ [K_-, K_+] = K_3, \tag{7}$$

which admits the ∞-dimensional UIR (6). Based on the spectrum of the values of the energy defined in (4-6), the Hamiltonian (2) could be rewritten in the following way $H = \hbar\omega K_+K_-$. From (6) a number operator is constructed given by

$$N = (1/2)(K_3 - \eta), \qquad N \,|\, \eta/2, n\rangle = n \,|\, \eta/2, n\rangle\,, \tag{8}$$

where the eigenstates of the number operator N constitute an ortonormal base for a Fock space. The existence of the dynamical algebra $\mathfrak{su}(1,1)$ associated to the PT potentials, permits the construction of generalized coherent states of Barut-Girardello type. These states are the eigenvectors of annihilation operator, $K_- \,|\, \eta/2, z\rangle = z \,|\, \eta/2, z\rangle$, where $z \in \mathbb{C}$, and η is a positive integer, and are defined by [5]

$$|\, \eta/2, z\rangle = \frac{1}{\sqrt{f(z)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\,(\eta)_n}}\,|\, \eta/2, n\rangle\,, \tag{9}$$

where $f(z) = \left\{\Gamma(\eta)\,|z|^{-(\eta-1)}\, I_{\eta-1}(2\,|z|)\right\}$, $(\eta)_n$ is Pochammer' symbol $(\eta)_n = \eta(\eta+1)\ldots(\eta+n-1)$, and I_ν is the modified Bessel function of first class. The distribution of probability to the discrete random variable n associted to (9) is

$$P_n(\eta/2, z) = f(z)^{-1}\frac{|z|^{2n}}{n!\,(\eta)_n}. \tag{10}$$

The Algorithm

Given a Diophantine equation with k unknowns,

$$D(x_1, \ldots, x_k) = 0, \tag{11}$$

based on the Algorithm of Kieu we provides the following quantum algorithm to decide whether this equation has any non-negative integer solution or not:

1. Construct or simulate a physical process in which a system initially begins from a state that is a direct product of k coherent states

$$|\,\psi(0)\rangle = \bigotimes_{i=1}^{k} |\,\eta_i/2, z_i\rangle\,. \tag{12}$$

   and in which the system is submitted to the action of a Hamiltonian $H_A(t)$ dependent of the time over the interval $[0, T]$, for a time T

$$H_A(t) = \left(1 - \frac{t}{T}\right) H_I + \frac{t}{T} H_D, \tag{13}$$

   with the initial Hamiltonian H_I and the final Hamiltonian H_D

$$H_I = \sum_{i=1}^{k} (K_{+_i} - z_i^*)(K_{-_i} - z_i), \qquad H_D = \left(D\left(N_1, \ldots, N_k\right)\right)^2.$$

2. Measure or estimate, by way of Schrödinger equation with the Hamiltonian $H_A(t)$, the maximum probability of finding the system in a particular multi-particle state in the chosen time T

$$P_{max}(T) = \max_{|\,\{n\}\rangle} |\,\langle\psi(T)\,|\,\{n\}\rangle\,|^2 = |\langle\psi(T)\,|\,\{n\}_0\rangle|^2\,, \tag{14}$$

   where $|\,\{n\}_0\rangle$ (which is direct product of k particular states, $\bigotimes_{i=1}^{k} \left|\,n_i^0\right\rangle$) possesses the maximum probability amongst the rest of the multi-particle states.

3. If $P_{\max}(T) \leq 1/2$, increase T and repeat the previous steps. If

$$P_{\max}(T) > 1/2 \tag{15}$$

   then $|\,\{n\}_0\rangle$ is the fundamental state of H_D (it is assumed that there is no spectral degeneration) and the following conclusion is obtained: $H_D\,|\,\{n\}_0\rangle = 0$, if and only if, equation (11) has a non-negative integer solution.

In order to satisfy the halting criterion (15), is necessary that $P_{max}(0) \leq 1/2$. For a $\eta/2$ fixed established by the quantum system there is infinities values of z that satisfy that condition, according to distribution on (10)

Limit Case: The Infinite Square Well

As a limited case of our hypercomputational algorithm *à la* Kieu on the Pöschl-Teller potentials, we obtain our hypercomputational algorithm on the infinite square well (ISW) presented previously [3, 4]. The explicit infinite-dimensional UIR of the dynamical algebra $\mathfrak{su}(1,1)$, of the Fock space associated to the UIR and of the actions of the creation and annihilation operators over the states of the Fock space, as well as the number operator and the Barut-Girardello coherent state corresponding to the ISW, are obtained by replacing $\eta = 3$ in the respective expressions of the PT potentials. This way we obtain our algorithm *à la* Kieu on the infinite square well.

Conclusions

It is inferred from what has been exposed that Kieu's algorithm consists of four basic parts: (i) Codification of the instance to resolve of Hilbert's tenth problem, (ii) Establishment of initial conditions, (iii) Evolution from an initial state to a final stage, (iv) Setting of halting criterion. Part (i) is founded upon a dynamical algebra associated with the physical referent applied in the description of the algorithm. Part (ii) is established based on the coherent states and the ladder operators associated to the dynamical algebra of physical system. Part (iii) is based upon an adiabatic quantum computation regarding unbounded Hamiltonians. Part (iv) requires certain properties from the initial state based on the distribution of the probability of the coherent states associated with the dynamical algebra. In the present work we have we have carried out a variation in the parts (i), (ii) and (iv) with respect to Kieu's algorithm.

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