Representations of Ordinal Numbers

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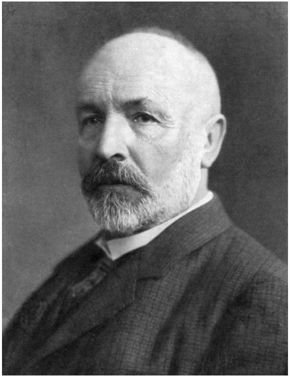
*Tutor

Ordinal numbers

Cantor



Cantor at early 20th century.*

Cantor defined ordinal numbers by two principles of generation and a first ordinal [Tiles 2004].

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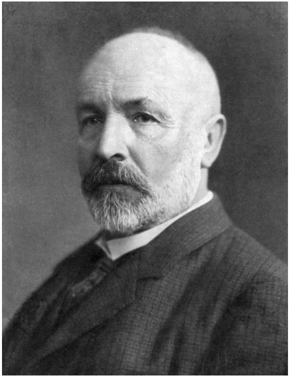
Ordinal numbers
Cantor



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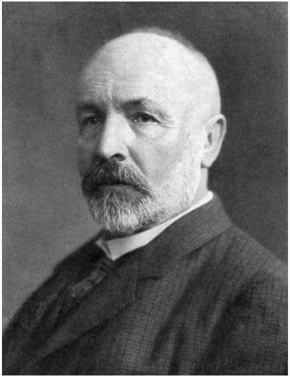

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- The successor of an ordinal number is an ordinal number.

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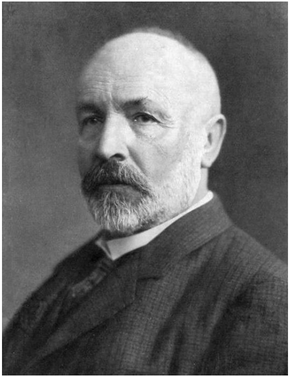

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Cantor defined ordinal numbers by two principles of generation and a first ordinal [Tiles 2004].

- 0 is the first ordinal number.
- The successor of an ordinal number is an ordinal number.
- The limit of an infinite increasing sequence of ordinals is an ordinal number.

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Example

Let's construct some ordinals using the previous rules.

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$$\omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots, \omega^{\omega^{\omega^{\omega^\omega}}}, \ldots, \epsilon_0, \ldots$$

Ordinal numbers

von Neumann Ordinals

von Neumann [1928] defined ordinals by:

Definition

An ordinal is a set α that satisfies:

- For every $y \in x \in \alpha$ it occurs that $y \in \alpha$. This is called a transitive property.
- The set α is well-ordered by the membership relationship.

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Remark

Observe that the definition is not recursive as Cantor's.

Ordinal numbers

Some von Neumann Ordinals

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$$1 := \{0\}$$
$$2 := \{0, 1\}$$

Ordinal numbers

Some von Neumann Ordinals

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\begin{aligned}
0 &:= \emptyset \\
1 &:= \{0\} \\
2 &:= \{0, 1\} \\
&\vdots \\
\omega &:= \{0, 1, 2, \ldots\} \\
\omega + 1 &:= \{0, 1, 2, \ldots, \omega\} \\
&\vdots
\end{aligned}
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It is important to see that it occurs that:

$$0 \in 1 \in 2 \in \ldots \omega \in \omega + 1 \in \ldots$$

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A countable ordinal is an ordinal whose cardinality is finite or denumerable.

Ordinal numbers

Countable Ordinals

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It is important to notice that the countable ordinals are the ordinals of the first and second class of Cantor.

Ordinal numbers

Hilbert Definition

Hilbert defined the natural and ordinal numbers using predicate logic [Hilbert 1925].

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$$\mathrm{Nat}(0)$$
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&\mathrm{On}(0) \\
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&\{\forall n[\mathrm{Nat}(n) \rightarrow \mathrm{On}(f(n))]\} \rightarrow \mathrm{On}(\lim(f(n))) \\
&\{P(0) \wedge \forall n[P(n) \rightarrow P(\mathrm{succ}(n))] \wedge \forall f \forall n[P(f(n)) \rightarrow P(\lim f)]]\} \\
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where Nat and On are propositional functions representing both numbers.

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Computable Ordinals

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The first countable ordinal that is non-computable is called $\omega_1^{\mathsf{CK}*}$.

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The computable ordinals are less than the countable ones, as there are less λ-terms than real numbers.

The first countable ordinal that is non-computable is called ω_1^{CK}*.
Furthermore, all non-countable ordinals are non-computable.

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Representations

Hardy

Hardy represented ordinals by sequences of natural numbers and defined two operations [Hardy 1904].

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- 1, $\mathbf{2}$, 3, ... $\to$ 1
- 2, 3, $\mathbf{4}$, ... $\to$ 2

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- **0**, 1, 2, ... $\rightarrow 0$
- 1, **2**, 3, ... $\rightarrow 1$
- 2, 3, **4**, ... $\rightarrow 2$

  ⋮
- **0**, 2, 4, 6 ... $\rightarrow \omega$
- 2, **4**, 6, 8 ... $\rightarrow \omega + 1$
- 4, 6, **8**, 10 ... $\rightarrow \omega + 2$

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- 2, **4**, 6, 8 ... $\to \omega + 1$
- 4, 6, **8**, 10 ... $\to \omega + 2$

  ⋮
- 0, 4, 8, 12, ... $\to \omega \cdot 2$
- 4, 8, 12, 16, ... $\to \omega \cdot 2 + 1$
- 8, 12, 16, 20, ... $\to \omega \cdot 2 + 2$

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(\omega + 1)_x &:= 2(x + 1) \\
(\omega + 2)_x &:= 2(x + 2) \\
&\vdots \\
(\omega \cdot n + k)_x &:= 2^n(x + k)
\end{aligned}
$$

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$$\frac{}{\mathsf{zero}_{\mathsf{o}} : \mathrm{On}} \qquad \frac{n : \mathrm{On}}{\mathsf{succ}_{\mathsf{o}}\ n : \mathrm{On}} \qquad \frac{f : \mathrm{Nat} \to \mathrm{On}}{\mathsf{lim}\ f : \mathrm{On}}$$

Remark

Martin-Löf's definition is analogous to Cantor and Hilbert's definition.

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Question

Which ordinal cannot be constructed by Martin-Löf's representation?

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Question

Which ordinal cannot be constructed by Martin-Löf's representation?

Is it possible to define, similarly, a ω_1^{ML}?

References I

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