Towards a Paraconsistent Type Theory

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Outline

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Intuitionistic logic

"Intuitionistic logic, sometimes more generally called constructive logic, refers to systems of symbolic logic that differ from the systems used for classical logic by more closely mirroring the notion of constructive proof... Formalized intuitionistic logic was originally developed by Arend Heyting to provide a formal basis for Brouwer's programme of intuitionism."
[Wikipedia contributors, 2019]

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Intuitionistic logic

A natural deduction system for Intuitionistic Logic (IL) ([van Dalen, 2013]):

$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\supset\mathrm{I}$	$\dfrac{A \supset B \quad A}{B}\supset\mathrm{E}$
$\dfrac{A \quad B}{A \wedge B}\wedge\mathrm{I}$	$\dfrac{A \wedge B}{A}\wedge\mathrm{E}_1 \quad \dfrac{A \wedge B}{B}\wedge\mathrm{E}_2$
$\dfrac{A}{A \vee B}\vee\mathrm{I}_1$ $\dfrac{B}{A \vee B}\vee\mathrm{I}_2$	$\dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\vee\mathrm{E}$

Table: Rules for propositional connectives in IL

Definition of negation: $\neg A \stackrel{\text{def}}{=} A \supset \bot$.

$$\frac{\bot}{A}\ \bot\mathsf{E}$$

Table: Bottom elimination rule

Intuitionistic logic

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Table: Bottom elimination rule

$\dfrac{A(x)}{\forall x.A(x)}\ \forall\text{I}$	$\dfrac{\forall x.A(x)}{A(t)}\ \forall\text{E}$
$\dfrac{A(t)}{\exists x.A(x)}\ \exists\text{I}$	$\dfrac{\exists x.A(x) \quad \begin{matrix}[A(y)] \\ \vdots \\ C\end{matrix}}{C}\ \exists\text{E}$

Table: Rules for quantifiers in IL

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- The following are not theorems of IL:
  - **Tertium Non Datur**: $A \vee \neg A$.
  - **Doble negation elimination**: $\neg\neg A \supset A$.
  - **Reductio ad adsurdum**: $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$.

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- Several intuitionistic theories, for instance Heyting arithmetic, satisfy the **existence property**: if $\exists x.A(x)$ is a theorem of the theory, then $A(t)$ is a theorem of the theory for some term t.

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- Several intuitionistic theories, for instance Heyting arithmetic, satisfy the **existence property**: if $\exists x.A(x)$ is a theorem of the theory, then $A(t)$ is a theorem of the theory for some term t.
- IL satisfies **substitution by equivalents**.

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Intuitionistic logic

The Brouwer-Heyting-Kolmogorov (BHK) interpretation (or proof interpretation) for IL ([Troelstra and van Dalen, 1988]):

- a proves $A \wedge B$ if a is a pair $\langle b, c \rangle$ such that b proves A and c proves B.
- a proves $A \vee B$ if a is a pair $\langle b, c \rangle$ such that b is a natural number and if $b = 0$ then c proves A, otherwise c proves B.
- a proves $A \supset B$ if a is a construction that converts any proof p of A into a proof $a(p)$ of B.
- no a proves $\perp$.

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- a proves $A \supset B$ if a is a construction that converts any proof p of A into a proof $a(p)$ of B.
- no a proves $\bot$.

In order to deal with the quantifiers, it is assumed that some domain D of objects is given.

- a proves $\forall x.A(x)$ if a is a construction such that, for each $b \in D$, $a(b)$ proves $A(\overline{b})$.
- a proves $\exists x.A(x)$ if a is a pair $\langle b, c \rangle$ such that $b \in D$ and c proves $A(\overline{b})$.

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Intuitionistic type theory

A brief historical review:

- (1908) Bertran Russel's proposed a ramified theory of types.
- (1920s) Leon Chwistek and Frank P. Ramsey proposed an unramified type theory, now known as theory of simple types (or simple theory of types).
- (1940) Alonso Church introduced the simply typed lambda-calculus.
- (1958) Haskell Curry establishes a correspondence between the simply typed lambda-calculus and the implicational fragment of intuitionistic logic.
- (1969) William A. Howard extended the correspondence to firts-order predicate logic, which is now known as the Curry-Howard correspondence.
- (1970s) Per Martin-Löf introduces several different versions of his theory of types.

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The name "intuitionistic type theory (ITT)" is somewhat ambiguous, but usually refers to a version (or a modified version) of Martin-Löf's type theory. Because of that, "intuitinistic type theory" and "Martin-Löf's type theory" are considered synonyms.

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ITT is based on:

- Martin-Löf's analysis of the notion of "judgement" (in mathematics).
- Martin-Löf's (intuitionistic) conception of the logical connectives.
- The Curry-Howard correspondence.
- A (not precisely defined) notion of "inductive definition".

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Intuitionistic type theory

ITT extends the BHK-interpretation "*to the more general setting of intuitionistic type theory and thus provides a general conception not only of what a constructive proof is, but also of what a constructive mathematical object is.*"
([Dybjer and Palmgren, 2016]).

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For Martin-Löf, "*A judgement is an act of knowledge, for instance asserting that something holds. When reasoning mathematically we are making a sequence of judgements about mathematical objects. One kind of judgement may be to state that some mathematical statement is true, another kind of judgement may be to state that something is a mathematical object, is a set, for instance. The logical rules give a method for producing correct judgements from earlier judgements.*" [Palmgren, 2013, p. 10]

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Martin-Löf's type theory have four basic forms of judgements ([Martin-Löf, 1984, p. 5–10]):

1. *A* **is a set** (abbreviated A : set).
2. *A* **and** *B* **are equal sets** (abbreviated A = B).
3. *a* **is an element of the set** *A* (abbreviated a : A).
4. *a* **and** *b* **are equal elements of the set** *A* (abbreviated a = b : A).

Intuitionistic type theory

Four different interpretations for Martin-Löf's judgements forms are ([Martin-Löf, 1984, p. 5]):

A : *set*	$a : A$	
A is a set	a is an element of the set A	A is non-empty
A is a proposition	a is a proof (construction) of proposition A	A is true
A is an intention (expectation)	a is a method of fulfilling (realizing) the intention (expectation) A	A is fulfillable (realizable)
A is a problem (task)	a is a method of solving the problem (doing the task) A	A is solvable

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Intuitionistic type theory

Defining a set (or a type) in Martin-Löf's type theory requires the following rules ([Martin-Löf, 1984, p. 24]):

- Formation: says that we can form a certain set from certain other sets or families of sets.
- Introduction: say what are the canonical elements (and equal canonical elements) of the set, thus giving its meaning.
- Elimination: shows how we may define functions on the set defined by the introduction rules.
- Equality/Computation: relate the introduction and elimination rules by showing how a function defined by means of the elimination rule operates on the canonical elements of the set which are generated by the introduction rules.

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Intuitionistic type theory

Definition of sets related with logical connectives ([Palmgren, 2013]):

Function set:

$\rightarrow$-formation $$\frac{A : set \quad B : set}{A \rightarrow B : set}$$	$\rightarrow$-introduction $$\frac{\begin{matrix}[x : A] \\ \vdots \\ b(x) : B\end{matrix}}{\lambda x.b(x) : A \rightarrow B}$$
$\rightarrow$-elimination $$\frac{b : A \rightarrow B \quad a : A}{Ap(b, a) : B}$$	$\rightarrow$-computation $$\frac{\begin{matrix} & [x : A] \\ & \vdots \\ \mathrm{a} : \mathrm{A} & b(x) : B\end{matrix}}{Ap(\lambda x.b(x), a) = b(a) : B}$$

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Intuitionistic type theory

Product of two sets:

$\times$-formation $$\frac{A : set \quad B : set}{A \times B : set}$$	$\times$-introduction $$\frac{a : A \quad b : B}{(a, b) : A \times B}$$
$\times$-elimination $$\frac{c : A \times B}{\pi_1(c) : A} \quad \frac{c : A \times B}{\pi_2(c) : B}$$	$\times$-computation $$\frac{a : A \quad b : B}{\pi_1((a, b)) = a : A}$$ $$\frac{a : A \quad b : B}{\pi_2((a, b)) = b : B}$$

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Disjoint union of two sets:

+-formation $\dfrac{A : set \qquad B : set}{A + B : set}$	+-introduction $\dfrac{a : A}{\mathrm{inl}(a) : A + B} \quad \dfrac{b : B}{\mathrm{inr}(b) : A + B}$
+-elimination $\dfrac{c : A + B \qquad \begin{matrix}[x : A] \\ \vdots \\ d(x) : C\end{matrix} \qquad \begin{matrix}[y : B] \\ \vdots \\ e(y) : C\end{matrix}}{D(c, (x)d(x), (y)e(y)) : C}$	+-computation $\dfrac{\mathrm{inl}(a) : A + B \qquad \begin{matrix}[x : A] \\ \vdots \\ d(x) : C\end{matrix} \qquad \begin{matrix}[y : B] \\ \vdots \\ e(y) : C\end{matrix}}{D(\mathrm{inl}(a), (x)d(x), (y)e(y)) = d(a) : C}$ $\dfrac{\mathrm{inr}(b) : A + B \qquad \begin{matrix}[x : A] \\ \vdots \\ d(x) : C\end{matrix} \qquad \begin{matrix}[y : B] \\ \vdots \\ e(y) : C\end{matrix}}{D(\mathrm{inr}(b), (x)d(x), (y)e(y)) = e(b) : C}$

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Product of a family of sets:

Π-formation $$\dfrac{A : set \qquad \begin{matrix}[x : A] \\ \vdots \\ B(x) : set\end{matrix}}{\Pi x : A.B(x) : set}$$	Π-introduction $$\dfrac{\begin{matrix}[x : A] \\ \vdots \\ b(x) : B(x)\end{matrix}}{\lambda x.b(x) : \Pi x : A.B(x)}$$
Π-elimination $$\dfrac{b : \Pi x : A.B(x) \qquad a : A}{Ap(b, a) : B(a)}$$	Π-computation $$\dfrac{a : A \qquad \begin{matrix}[x : A] \\ \vdots \\ b(x) : B(x)\end{matrix}}{Ap(\lambda x.b(x), a) = b(a) : B(a)}$$

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Disjoint union of a family of sets:

Σ-formation

$$\frac{A : set \qquad \begin{matrix}[x : A] \\ \vdots \\ B(x) : set\end{matrix}}{\Sigma x : A.B(x) : set}$$

Σ-introduction

$$\frac{a : A \qquad b : B(a)}{(a, b) : \Sigma x : A.B(x)}$$

Σ-elimination

$$\frac{c : \Sigma x : A.B(x) \qquad \begin{matrix}[x : A, y : B(x)] \\ \vdots \\ d(x, y) : C((x, y))\end{matrix}}{E(c, (x, y)d(x, y)) : C(c)}$$

Σ-computation

$$\frac{a : A \quad b : B(a) \qquad \begin{matrix}[x : A, y : B(x)] \\ \vdots \\ d(x, y) : C((x, y))\end{matrix}}{E((a, b), (x, y)d(x, y)) = d(a, b) : C((a, b))}$$

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Empty set:

N_0-formation	N_0-elimination
$$\frac{}{N_0 : set}$$	$$\frac{c : N_0 \quad C : set}{R_0(c) : C}$$

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The Curry-Howard correspondence for ITT:

Logic	Type theory
Proposition	Type
Connective	Type constructor
Implication	Function type
Conjunction	Product of two types
Disjunction	Disjoint union of two types
For all	Product of a family of types
Exists	Disjoint union of a family of types
Absurdity	Empty type
Proof	Term
Provability	Inhabitation

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1. Intuitionistic logic
2. Intuitionistic type theory
3. Nelson's paraconsistent logic
4. Towards a paraconsistent type theory

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Some objections against IL:

"*... definitions of constructiveness other than that advocated by the intuitionists are conceivable. For that matter, even the small number of actual intuitionists do not completely agree about the delimination of the constructive.*" [Heyting, 1971, p. 10]

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"Serious objections against the use of negation in mathematics have been raised by Griss [...]. Though agreeing completely with Brouwer's basic ideas on the nature of mathematics, he contends that every mathematical notion has its origin in a mathematical construction, which can actually be carried out; if the construction is impossible, then the notion cannot be clear." [Heyting, 1971, p. 124]

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"*The justification [of the ex falso rule] in terms of constructions is not universally accepted, e.g. [Johansson, 1936] rejected the rule and formulated his so-called minimal logic, which has the same rules of intuitionistic logic with deletion of the ex falso rule.*" [van Dalen, 2002, p. 12]

Nelson's paraconsistent logic

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Some proposals to overcome the objections against IL:

- **Georg F. C. Griss**: eliminate negation from IL (negationless constructive mathematics).
- **Ingebrigt Johansson**: eliminate the *ex falso* rule from IL (minimal logic).
- **David Nelson**: define constructive rules for negation (Nelson's logic with strong (or constructive) negation).

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Nelson's paraconsistent logic

A natural deduction system for Nelson's Logic N (also known as N3) is obtained by adding to the system for IL the rules ([Prawitz, 1965, p. 97]):

$\dfrac{A}{\neg\neg A}\ \neg\neg\mathrm{I}$	$\dfrac{\neg\neg A}{A}\ \neg\neg\mathrm{E}$
$\dfrac{A \quad \neg B}{\neg(A \supset B)}\ \neg{\supset}\mathrm{I}$	$\dfrac{\neg(A \supset B)}{A}\ \neg{\supset}\mathrm{E}_1 \quad \dfrac{\neg(A \supset B)}{\neg B}\ \neg{\supset}\mathrm{E}_2$
$\dfrac{\neg A}{\neg(A \wedge B)}\ \neg{\wedge}\mathrm{I}_1$ $\dfrac{\neg B}{\neg(A \wedge B)}\ \neg{\wedge}\mathrm{I}_2$	$\dfrac{\neg(A \wedge B) \quad \begin{matrix}[\neg A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[\neg B]\\ \vdots \\ C\end{matrix}}{C}\ \neg{\wedge}\mathrm{E}$
$\dfrac{\neg A \quad \neg B}{\neg(A \vee B)}\ \neg{\vee}\mathrm{I}$	$\dfrac{\neg(A \vee B)}{\neg A}\ \neg{\vee}\mathrm{E}_1 \quad \dfrac{\neg(A \vee B)}{\neg B}\ \neg{\vee}\mathrm{E}_2$

Table: Rules for negation of propositional connectives in N

Nelson's paraconsistent logic

$\dfrac{\neg A(t)}{\neg\forall x.A(x)}\ \neg\forall\text{I}$	$\dfrac{\neg\forall x.A(x) \qquad \begin{matrix}[\neg A(y)] \\ \vdots \\ C\end{matrix}}{C}\ \neg\forall\text{E}$
$\dfrac{\neg A(x)}{\neg\exists x.A(x)}\ \neg\exists\text{I}$	$\dfrac{\neg\exists x.A(x)}{\neg A(t)}\ \neg\exists\text{E}$

Table: Rules for negation of quantifiers in N

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Nelson's paraconsistent logic

$\dfrac{\neg A(t)}{\neg\forall x.A(x)}\,\neg\forall\mathrm{I}$	$\dfrac{\neg\forall x.A(x) \qquad \begin{array}{c}[\neg A(y)]\\ \vdots \\ C\end{array}}{C}\,\neg\forall\mathrm{E}$
$\dfrac{\neg A(x)}{\neg\exists x.A(x)}\,\neg\exists\mathrm{I}$	$\dfrac{\neg\exists x.A(x)}{\neg A(t)}\,\neg\exists\mathrm{E}$

Table: Rules for negation of quantifiers in N

Intuitionistic and strong negation are connected by the additional rule:

$\dfrac{A \qquad \neg A}{\bot}\,\bot\mathrm{I}$

Table: Bottom introduction rule

Under a logic L:

- A theory Γ is contradictory if there exists a formula A such that $\Gamma \vdash_L A$ and $\Gamma \vdash_L \neg A$.
- A theory Γ is trivial if $\Gamma \vdash_L A$ for every formula A.

A paraconsistent logic is a logic that admits contradictory but non-trivial theories.

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A paraconsistent logic is a logic that admits contradictory but non-trivial theories.

IL and N are not paraconsistent logics.

Nelson's logic $\mathbf{N}^-$ (also known as N4), introduced in [Almukdad and Nelson, 1984]), can be defined by removing the bottom rules from N. The logic $\mathbf{N}^-$ **is paraconsistent**.

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- The following are not theorems of N^-:
  - **Non-contradiction**: $\neg(A \wedge \neg A)$.
  - **Ex falso sequitur quodlibet**: $\neg A \supset (A \supset B)$.
  - **Law of contradiction**: $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.
  - **Reductio ad adsurdum**: $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$.

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  - **Reductio ad adsurdum**: $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$.
- The following are theorems of N^-:
  - **Doble negation**: $(\neg\neg A \supset A) \wedge (A \supset \neg\neg A)$.
  - **De Morgan laws**. $(\neg(A \wedge B) \supset (\neg A \vee \neg B)) \wedge ((\neg A \vee \neg B) \supset \neg(A \wedge B))$

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- N^- satisfies the **disjunction property**: $\vdash_{N^-} A \vee B$ iff $\vdash_{N^-} A$ or $\vdash_{N^-} B$.

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- N^- satisfies the **disjunction property**: $\vdash_{N^-} A \vee B$ iff $\vdash_{N^-} A$ or $\vdash_{N^-} B$.
- N^- satisfies the **constructive falsity principle**: $\vdash_{N^-} \neg(A \wedge B)$ iff $\vdash_{N^-} \neg A$ or $\vdash_{N^-} \neg B$.

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- N^- satisfies the **constructive falsity principle**: $\vdash_{N^-} \neg(A \wedge B)$ iff $\vdash_{N^-} \neg A$ or $\vdash_{N^-} \neg B$.
- N^- does not satisfy **substitution by equivalents** (however substitution is valid for equivalent formulas whose negations are also equivalents).

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Nelson's paraconsistent logic

Extension of the BHK-interpretation, where refutation is a primitive notion and a construction c proves $\neg A$ iff c refutes A ([López-Escobar, 1972]):

- c refutes $\neg A$ iff c proves A.
- c refutes $A \wedge B$ iff $c = (i, d)$ with i either 0 or 1 and if $i = 0$, then d refutes A and if $i = 1$ then d refutes B.
- c refutes $A \vee B$ iff $c = (d, e)$ and d refutes A and e refutes B.
- c refutes $A \supset B$ iff $c = (d, e)$ and d proves A and e refutes B.
- c refutes $\forall x.A(x)$ iff $c = (a, d)$ and d refutes $A(a)$.
- c refutes $\exists x.A(x)$ iff c is a general method of construction such that given any individual (i.e construction) a from the species under consideration, $c(a)$ (i.e. c applied to a) refutes $A(a)$.

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Outline

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Towards a paraconsistent type theory

Main objective: construct a type theory based on Nelson's paraconsistent logic N^-.

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Towards a paraconsistent type theory

Main objective: construct a type theory based on Nelson's paraconsistent logic N^-.

Related works:

- [Wansing, 1993]: introduced a typed lambda-calculus (λ^C) where types are the propositional formulas of N^- and rules are based on the rules of N^-. A formulas-as-types correspondence and a semantics for λ^C is provided. Assignation of types is not unique and issues like normalization are left open.
- [Kamide, 2010]: a different typed lambda calculus for the propositional fragment of N^- is provided and strong normalization for this calculus is proven.

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Towards a paraconsistent type theory

A proposal of a paraconsistent theory of types (PTT): add opposite types to ITT, including introduction, elimination and computation rules for each type constructor.

$^{-}$-formation	
$\dfrac{A : set}{\overline{A} : set}$	
$^{-}$-introduction	$^{-}$-elimination
$\dfrac{a : A}{a : \overline{\overline{A}}}$	$\dfrac{a : \overline{\overline{A}}}{a : A}$
$\dfrac{a : A \qquad b : \overline{B}}{(a, b) : \overline{A \rightarrow B}}$	$\dfrac{c : \overline{A \rightarrow B}}{\pi_1(c) : A}$ $\dfrac{c : \overline{A \rightarrow B}}{\pi_2(c) : \overline{B}}$

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$\overline{}$-introduction	$\overline{}$-elimination
$\dfrac{a : \overline{A}}{\mathrm{inl}(a) : \overline{A \times B}}$ $\dfrac{b : \overline{B}}{\mathrm{inr}(b) : \overline{A \times B}}$	$\dfrac{c : \overline{A \times B} \quad \begin{matrix}[x : \overline{A}] \\ \vdots \\ d(x) : C\end{matrix} \quad \begin{matrix}[y : \overline{B}] \\ \vdots \\ e(y) : C\end{matrix}}{D(c, (x)d(x), (y)e(y)) : C}$
$\dfrac{a : \overline{A} \quad b : \overline{B}}{(a, b) : \overline{A + B}}$	$\dfrac{c : \overline{A + B}}{\pi_1(c) : \overline{A}}$ $\dfrac{c : \overline{A + B}}{\pi_2(c) : \overline{B}}$

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Towards a paraconsistent type theory

¯-introduction	¯-elimination
$\dfrac{a : A \quad b : \overline{B(a)}}{(a, b) : \overline{\Pi x : A.B(x)}}$	$\dfrac{c : \overline{\Pi x : A.B(x)} \quad \begin{matrix}[x : A, y : \overline{B(x)}] \\ \vdots \\ d(x, y) : C((x, y))\end{matrix}}{E(c, (x, y)d(x, y)) : C(c)}$
$\dfrac{\begin{matrix}[x : A] \\ \vdots \\ b(x) : \overline{B(x)}\end{matrix}}{\lambda x.b(x) : \overline{\Sigma x : A.B(x)}}$	$\dfrac{b : \overline{\Sigma x : A.B(x)} \quad a : A}{Ap(b, a) : \overline{B(a)}}$

Note: ¯-computation rules are defined just in the same way that in ITT.

Towards a paraconsistent type theory

The Curry-Howard correspondence for PTT:

Logic	Type theory
Proposition	Type
Connective	Type constructor
Implication	Function type
Conjunction	Product of two types
Disjunction	Disjoint union of two types
For all	Product of a family of types
Exists	Disjoint union of a family of types
Negation	Opposite type
Proof/Refutation	Term
Provability/Refutability	Inhabitation

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Some properties of PTT:

- Tolerance to contradictions.
- No uniqueness of types.
- Strong normalization.

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