Kripke Semantics

A Semantic for Intuitionistic Logic

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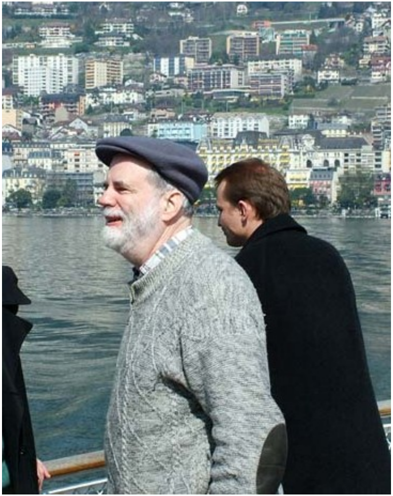
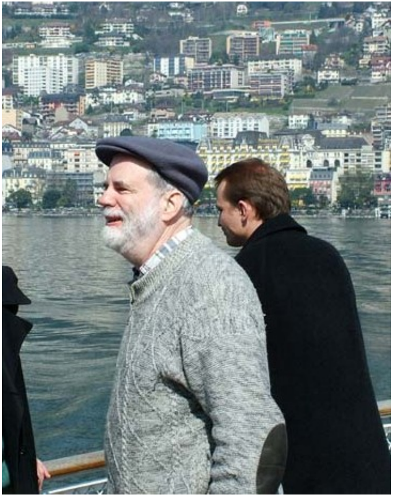
Saul Kripke

- He was born on November 13, 1940 (age 75)
- Philosopher and Logician
- Emeritus Professor at Princeton University
- In logic, his major contributions are in the field of Modal Logic



Saul Kripke

- In Modal Logic, we attributed to him the notion of *Possible Worlds*
- Its notable ideas
  - Kripke structures
  - Rigid designators
  - Kripke semantics



Kripke Semantics

The study of semantic is the study of *the truth*

- Kripke semantics is one of many (see for instance (Moschovakis, 2015)) semantics for *intuitionistic* logic
- It tries to capture different possible *evolutions* of the world over time
- The abstraction of *a world* we call a *Kripke structure*
- Proof rules of intuitionistic logic are *sound* with respect to krikpe structures

Derivation (proof) rules of the $\wedge$ connective

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \wedge\text{-}\mathit{intro}$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \wedge\text{-}\mathit{elim}_1$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi} \wedge\text{-}\mathit{elim}_2$$

Intuitionistic Logic

Derivation (proof) rules of the $\vee$ connective

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-}\textit{intro}_1$$

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \sigma \qquad \Gamma, \psi \vdash \sigma}{\Gamma \vdash \sigma}\ \vee\text{-}\textit{elim}$$

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-}\textit{intro}_2$$

Intuitionistic Logic

Derivation (proof) rules of the $\to$ connective

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \to\text{-}\textit{intro} \qquad\qquad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi} \to\text{-}\textit{elim}$$

Intuitionistic Logic

Derivation (proof) rules of the $\neg$ connective where $\neg\varphi \equiv \varphi \to \bot$

$$\frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ \neg\text{-}\textit{intro} \qquad\qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \textit{explosion}$$

Intuitionistic Logic

Other derivation (proof) rules

$$\frac{}{\Gamma \vdash \top}\ \textit{unit} \qquad \frac{}{\Gamma, \varphi \vdash \varphi}\ \textit{assume} \qquad \frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\ \textit{weaken}$$

Classical Logic

The list of derivation rules are the same above plus the following rule

$$\frac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}\ RAA$$

Kripke Model

Def. A *Kripke model* is a quadruple $\mathcal{K} = \langle K, \Sigma, C, D \rangle$ where

- K is a (non-empty) *partially ordered set*
- C is a function defined on the constants of L
- D is a set-valued function on K
- Σ is a function on K such that
  - $C(c) \in D(l)$ for all $k \in K$
  - $D(k) \neq \emptyset$ for all $k \in K$
  - $\Sigma(k) \subset A_{t_k}$ for all $k \in K$

where A_{t_k} is the set of all atomic sentences of L with constants for the elements of $D(k)$. (See the full description (van Dalen, 2004) or review a short description on (Moschovakis, 2015))

Kripke Model (Cont.)

D and Σ satisfy the following conditions:

(i) $k \leq l \Rightarrow D(k) \subseteq D(l)$

(ii) $\perp \notin \Sigma(k)$, for all k

(iii) $k \leq l \Rightarrow \Sigma(k) \subseteq \Sigma(l)$

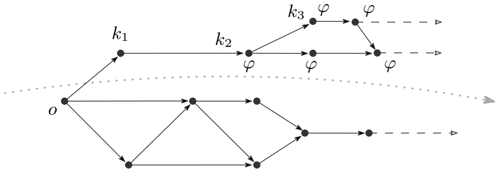


Figure: A Kripke model example

Lemma 1

Lemma

$\Sigma(k) \subseteq Sent_k$, $\Sigma(k)$ *satisfies*[1]*:*

(i) $\varphi \vee \psi \in \Sigma(k) \Leftrightarrow \varphi \in \Sigma(k)$ *or* $\psi \in \Sigma(k)$

(ii) $\varphi \wedge \psi \in \Sigma(k) \Leftrightarrow \varphi \in \Sigma(k)$ *and* $\psi \in \Sigma(k)$

(iii) $\varphi \rightarrow \psi \in \Sigma(k) \Leftrightarrow$ *for all* $l \geq k$ $(\varphi \in \Sigma(l) \Rightarrow \psi \in \Sigma(l))$

(iv) $\exists x \varphi(x) \in \Sigma(k) \Leftrightarrow$ *there is an* $a \in \mathcal{D}(k)$ *such that* $\varphi(\bar{a}) \in \Sigma(k)$

(v) $\forall x \varphi(x) \in \Sigma(k) \Leftrightarrow$ *for all* $l \geq k$ *and* $a \in \mathcal{D}(l)$ $\varphi(\bar{a}) \in \Sigma(l)$

Proof.

Immediate. (Also see (van Dalen, 2004, p. 165))

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[1] Set of all sentences with parameters in $\mathcal{D}(k)$.

Notation $k \Vdash \varphi$

Notation

We write $k \Vdash \varphi$ for $\varphi \in \Sigma(k)$ to say "k forces φ"

Using this notation, we can reformulate on terms of $\Vdash$. For instance look at the last two items

(iv) $k \Vdash \exists x \varphi(x) \Leftrightarrow$ there is an $a \in \mathcal{D}(k)$ such that $k \Vdash \varphi(\bar{a})$

(v) $k \Vdash \forall x \varphi(x) \Leftrightarrow$ for all $l \geq k$ and $a \in \mathcal{D}(l)$ $l \Vdash \varphi(\bar{a})$

Corollary

Corollary

(i) $k \Vdash \neg\varphi \Leftrightarrow$ *for all* $l \geq k, l \nVdash \varphi$

(ii) $k \Vdash \neg\neg\varphi \Leftrightarrow$ *for all* $l \geq k$*, there exists a* $p \geq l$ *such that* $(p \Vdash \varphi)$

Proof.

(i) $k \Vdash \neg\varphi \Leftrightarrow k \Vdash \varphi \to \bot \Leftrightarrow$ for all $l \geq k$ $(l \Vdash \varphi \Rightarrow l \Vdash \bot)$, for all $l \geq k$ $(l \nVdash \varphi)$

(ii) White-board

Monotonicity Lemma

Lemma

$k \leq l,\ k \Vdash \varphi \Rightarrow l \Vdash \varphi$

Proof.

- If φ is an atom, we are done by definition above
- If φ is $\varphi_1 \vee \varphi_2$
- Rest. White-board
- If φ is $\forall x \varphi_1(x)$, then let $k \Vdash \forall x \varphi_1(x)$ and $l \geq k$ Suppose $p \geq l$ and $a \in D(p)$, then, since $p \geq k$, $p \Vdash \varphi_1(\bar{a})$ Therefore, $l \Vdash \forall x \varphi_1(x)$

Example 1

In the bottom node k_0 no atoms are known, in the node k_1 only φ is known

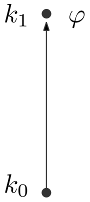

Figure: diagram example 1

$$k_0 \nVdash \varphi \vee \neg\varphi$$

Example 2

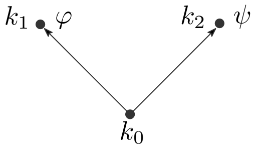

Figure: diagram example 2

$$k_0 \nVdash \neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$$

Example 3

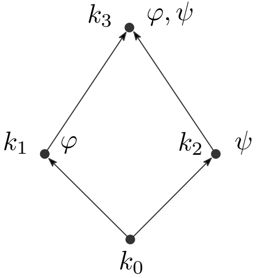

Figure: diagram example 3

$$k_0 \nVdash (\psi \to \varphi) \to (\neg\psi \lor \varphi)$$

Example 4

In the bottom node the following implications are forced:

$\varphi_2 \to \varphi_1$, $\varphi_3 \to \varphi_2$, $\varphi_3 \to \varphi_1$

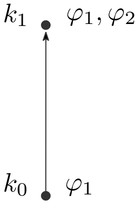

Figure: diagram example 4

$$
\begin{aligned}
k_0 \nVdash & (\varphi_1 \leftrightarrow \phi_2) \\
& \vee (\varphi_2 \leftrightarrow \phi_3 \\
& \vee (\varphi_1 \leftrightarrow \phi_3)
\end{aligned}
$$

Example 5

Note that in this example, $D(k_0) = \{0\}$ and $D(k_1) = \{0, 1\}$

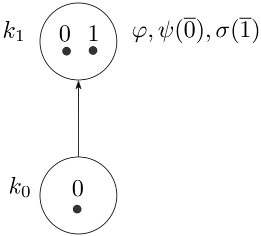

Figure: diagram example 5

$$k_0 \nVdash (\varphi \to \exists x \sigma(x)) \to \exists x(\varphi \to \sigma(x))$$

Example 6

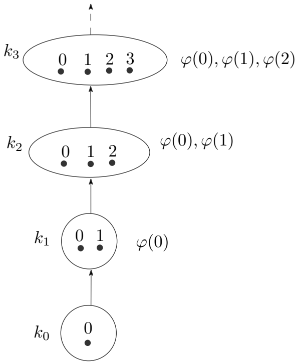

Figure: diagram example 6

$$k_0 \nVdash \forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$$

Theorem

(Soundness Theorem) $\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \Vdash \varphi$

Proof (See (van Dalen, 2004))
Use induction on the derivation $\mathcal{D}$ of φ from Γ. We will abbreviate "$k \Vdash \psi(\vec{a})$ for all $\psi \in \Gamma$" by "$k \Vdash \Gamma(\vec{a})$". The model $\mathcal{K}$ is fixed in the proof

(1) $\mathcal{D}$ consists of just φ, then obviously $k \Vdash \Gamma(\vec{a}) \;\Rightarrow\; k \Vdash \varphi(\vec{a})$ for all k and $(\vec{a}) \in \mathcal{D}(k)$

(2) $\mathcal{D}$ ends with application of a derivation rule

- ($\wedge$I) Induction hypothesis: $\forall k \forall \vec{a} \in \mathcal{D}(k)(k \Vdash \Gamma(\vec{a}) \;\Rightarrow\; k \Vdash \varphi_i(\vec{a}))$, for $i = 1, 2$. Now choose a $k \in K$ and $\vec{a} \in \mathcal{D}(k)$ such that $k \Vdash \Gamma(\vec{a})$, then $k \Vdash \varphi_1(\vec{a})$ and $k \Vdash \varphi_2(\vec{a})$, so $k \Vdash (\varphi_1 \wedge \varphi_2)(\vec{a})$
- ($\wedge$E) Immediate

($\vee$I) Immediate.

($\vee$E) Induction hypothesis: $\forall k(k \Vdash \Gamma \Rightarrow k \Vdash \varphi \vee \psi)$, $\forall k(k \Vdash \Gamma\, \varphi \Rightarrow k \Vdash \sigma)$, $\forall k(k \Vdash \Gamma\, \psi \Rightarrow k \Vdash \sigma)$. Now let $k \Vdash \Gamma$, then by the ind.hyp. $k \Vdash \phi \vee \psi$, so $k \Vdash \varphi$ or $k \Vdash \psi$). In the first case $k \Vdash \Gamma, \varphi$, so $k \Vdash \sigma$. In the second case $k \Vdash \Gamma, \psi$, so $k \Vdash \sigma$. In both cases $k \Vdash \sigma$, so we are done

($\rightarrow$I) Induction hypothesis: $(\forall k)(\forall \vec{a} \in \mathcal{D}(k))(k \Vdash \Gamma(\vec{a}), \varphi(\vec{a}) \Rightarrow k \Vdash \psi(\vec{a}))$. Now let $k \Vdash \Gamma(\vec{a})$ for some $\vec{a} \in \mathcal{D}(k)$. We want to show $k \Vdash (\varphi \rightarrow \psi)(\vec{a})$, so let $l \geq k$ and $l \Vdash \varphi(\vec{a})$. By monotonicity $l \Vdash \Gamma(\vec{a})$, and $\vec{a} \in \mathcal{D}(l)$, so ind. hyp. tell us that $l \Vdash \psi(\vec{a})$. Hence $\forall l \geq k(l \Vdash \varphi(\vec{a}) \Rightarrow l \Vdash \psi(\vec{a}))$, so $k \Vdash (\varphi \rightarrow \psi)(\vec{a})$

($\rightarrow$E) Immediate

($\bot$) Induction hypothesis: $\forall k(k \Vdash \Gamma \Rightarrow k \Vdash \bot)$. Since, evidently, no k can force Γ, $\forall k(k \Vdash \Gamma \Rightarrow k \Vdash \varphi)$ is correct

($\forall$I) The free variables in Γ are $\vec{x}$ and z does not occur in the sequence $\vec{x}$. Induction hypothesis: $(\forall k)(\forall \vec{a}, b \in \mathcal{D}(k))(k \Vdash \Gamma(\vec{a} \Rightarrow k \Vdash \varphi(\vec{a}, b))$. Now let $k \Vdash \Gamma(\vec{a})$ for some $\vec{a} \in \mathcal{D}(k)$, we must show $k \Vdash \forall z\varphi(\vec{a}, z)$. So let $l \geq k$ and $b \in \mathcal{D}(l)$. By monotonicity $l \Vdash \Gamma(\vec{a})$ and $\vec{a} \in \mathcal{D}(l)$, so by the ind. hyp. $l \Vdash \varphi(\vec{a}, b)$. This shows $(\forall l \geq k)(\forall b \in \mathcal{D}(l))(l \Vdash \varphi(\vec{a}, b))$, and hence $k \Vdash \forall z\varphi(\vec{a}, z)$

($\forall$E) Immediate

($\exists$I) Immediate

($\exists$E) Induction hypothesis:
$(\forall k)(\forall \vec{a} \in \mathcal{D}(k))(k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \exists z \varphi(\vec{a}, z))$ and
$(\forall k)(\forall \vec{a}, b \in \mathcal{D}(k))(k \Vdash \varphi(\vec{a}, b),\ k \Vdash \Gamma(\vec{a}) \Rightarrow k \Vdash \sigma(\vec{a}))$. Here the variables in Γ and σ are $\vec{x}$, and z does not occur in the sequence $\vec{x}$. Now let $k \Vdash \Gamma(\vec{a})$, for some $\vec{a} \in \mathcal{D}(k)$, then $k \Vdash \exists z \varphi(\vec{a}, z)$. So let $k \Vdash \varphi(\vec{a}, b)$ for some $b \in \mathcal{D}(k)$. By the induction hypothesis $k \Vdash \sigma(\vec{a})$

Theorem

(Completeness Theorem) $\Gamma \vdash \varphi \;\Leftrightarrow\; \Gamma \Vdash \varphi$ *(Γ and φ closed)*

Proof. (See (van Dalen, 2004) for the lemma mention below)
We have already shown $\Rightarrow$. For the converse we assume $\Gamma \nvdash \varphi$ and apply Lemma 6.3.9, which yields a contradiction.

References

[1] van Dalen, Dirk. *Logic and Structure*. 4th ed. Springer, 2004.

[2] Moschovakis, Joan. *Intuitionistic Logic, The Stanford Encyclopedia of Philosophy*, Edward N. Zalta, 2015.