Consistency of a Programming Logic for a Version of PCF Using Domain Theory

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Logic and Computation Seminar
Universidad EAFIT
5 April, 3 May 2013

A Core Functional Programming Language

Plotkin's PCF language

Theoretical Computer Science 5 (1977) 223–255.

LCF CONSIDERED AS A PROGRAMMING LANGUAGE

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Communicated by Robin Milner
Received July 1975

PCF Features [Plotkin 1977]

- Typed λ-calculus

Starting with a collection $\mathscr{L}$ of constants, each having a fixed type, and denumerably many variables α_i^σ $(i \geqslant 0)$ of each type, the *$\mathscr{L}$-terms* are given by the rules:

(1) Every variable α_i^σ is an $\mathscr{L}$-term of type σ.

(2) Every constant of type σ is an $\mathscr{L}$-term of type σ.

(3) If M and N are $\mathscr{L}$-terms of types $(\sigma \to \tau)$ and σ respectively then (MN) is an $\mathscr{L}$-term of type τ.

(4) If M is an $\mathscr{L}$-term of type τ then $(\lambda\alpha_i^\sigma M)$ is one of type $(\sigma \to \tau)$ $(i \geqslant 0)$.

PCF Features [Plotkin 1977]

- Basic data types: Natural numbers and Booleans

All languages, $\mathscr{L}$, considered include $\mathscr{L}_0$, the set of *standard* constants. These, together with their types, are:

$$tt : o,$$

$$ff : o,$$

$$\supset_\iota : (o, \iota, \iota, \iota),$$

$$\supset_o : (o, o, o, o),$$

$$Y_\sigma : ((\sigma \to \sigma) \to \sigma) \text{ (one for each } \sigma).$$

Generally we will be interested in a language $\mathscr{L}_A$ for arithmetic which also has:

$$k_n : \iota \text{ (one for each integer } n \geqslant 0),$$

$$(+1) : (\iota \to \iota),$$

$$(-1) : (\iota \to \iota),$$

$$Z : (\iota \to o).$$

A Programming Logic

Logical Theory of Constructions (LTC)

[Bove, Dybjer and Sicard-Ramírez 2009]

LTC = type-free version of PCF

(terms, conversion and discrimination rules)

+ first-order logic

+ inductive predicates (not considered in this talk)

LTC-Terms

Terms

$t ::= x$	variable
$\mid t \cdot t$	application
$\mid \lambda x.t$	λ-abstraction
$\mid \mathsf{fix}\ x.t$	fixed-point operator
$\mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}$	Boolean constants
$\mid 0 \mid \mathsf{succ} \mid \mathsf{pred} \mid \mathsf{iszero}$	natural number constants

Convention

The binary application function symbol $\cdot$ is left-associative.

LTC-Formulae

Formulae

$A ::= \top \mid \bot$	truth, falsehood
$\mid A \Rightarrow A \mid A \wedge A \mid A \vee A$	binary logical connectives
$\mid \forall x.A \mid \exists x.A$	quantifiers
$\mid t = t$	equality
$\mid P(t, \ldots, t)$	predicate

Abbreviations

$$\neg A \stackrel{\text{def}}{=} A \Rightarrow \bot,$$

$$t \neq t' \stackrel{\text{def}}{=} \neg(t = t').$$

Conversion and Discrimination Rules of LTC

Conversion rules

$$\begin{aligned}
\forall t\ t'.\ \mathsf{if} \cdot \mathsf{true} \cdot t \cdot t' &= t, \\
\forall t\ t'.\ \mathsf{if} \cdot \mathsf{false} \cdot t \cdot t' &= t', \\
\mathsf{pred} \cdot 0 &= 0, \\
\forall t.\ \mathsf{pred} \cdot (\mathsf{succ} \cdot t) &= t, \\
\mathsf{iszero} \cdot 0 &= \mathsf{true}, \\
\forall t.\ \mathsf{iszero} \cdot (\mathsf{succ} \cdot t) &= \mathsf{false}, \\
\forall t\ t'.\ (\lambda x.t) \cdot t' &= t[x := t'], \\
\forall t.\ \mathsf{fix}\ x.t &= t[x := \mathsf{fix}\ x.t],
\end{aligned}$$

where $t[x := t']$ is the capture-free substitution of x for t' in t.

Conversion and Discrimination Rules of LTC

Discrimination rules

$$\mathsf{true} \neq \mathsf{false},$$
$$\forall t.\ 0 \neq \mathsf{succ} \cdot t.$$

LTC Consistency

How we know that LTC is a consistent theory?

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Standard answer: To build a model for LTC
[Chang and Keisler 1992, theorem 1.3.21.]

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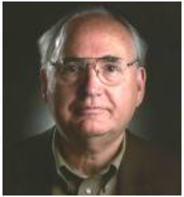
How we know that LTC is a consistent theory?

Standard answer: To build a model for LTC
[Chang and Keisler 1992, theorem 1.3.21.]
$\Rightarrow$ domain model for LTC

Introduction to Domain Theory

Motivation: Does λ-calculus have models?



"Historically my first model for the λ-calculus was discovered in 1969 and details were provided in Scott (1972) (written in 1971)." [Scott 1980, p. 226.]

Non-standard definitions

pre-domain, domain, complete partial order (cpo), ω-cpo, bottomless ω-cpo, Scott's domain, ...

Convention

domain $\equiv$ ω-complete partial order

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Partially Ordered Sets

Definition (Partially ordered set)

A partially ordered set (poset) $(D, \sqsubseteq)$ is a set D on which the binary relation $\sqsubseteq$ satisfies the following properties:

$$\forall x.\ x \sqsubseteq x \qquad \text{(reflexive)}$$

$$\forall x\ y\ z.\ x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z \qquad \text{(transitive)}$$

$$\forall x\ y.\ x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y \qquad \text{(antisymmetry)}$$

Partially Ordered Sets

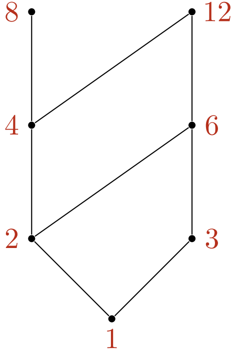
Examples

- $(\mathbb{Z}, \leq)$ is a poset.
- Let $a, b \in \mathbb{Z}$ with $a \neq 0$. The divisibility relation is defined by $a \mid b \stackrel{\text{def}}{=} \exists c\, (ac = b)$. Then $(\mathbb{Z}^+, |)$ is a poset.
- $(P(A), \subseteq)$ is a poset.

Partially Ordered Sets

Example

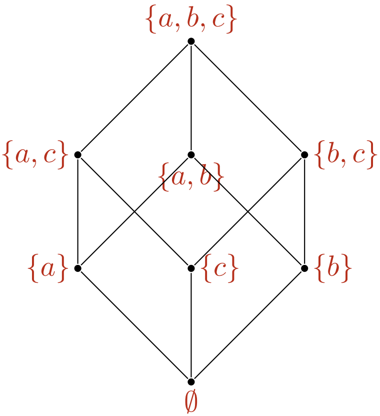
Hasse diagram for the poset $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.



Partially Ordered Sets

Example

Hasse diagram for the poset $(\{a, b, c\}, \subseteq)$.



Monotone Functions

Definition (Monotone function)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be two posets. A function $f : D \to D'$ is monotone if

$$\forall x\ y.\ x \sqsubseteq y \Rightarrow f(x) \sqsubseteq' f(y).$$

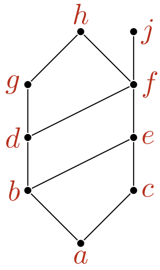
Notable Elements

Definition (Upper bound)

Let $(D, \sqsubseteq)$ be a poset and let $A \subseteq D$. Let $u \in D$ be an element such that $a \sqsubseteq u$ for all elements $a \in A$, then u is an upper bound of A.

Examples

- $A = \{a, b, c\}$
  Upper bounds: $\{e, f, j, h\}$
- $A = \{j, h\}$
  No upper bounds.
- $A = \{a, c, d, f\}$
  Upper bounds: $\{f, h, j\}$



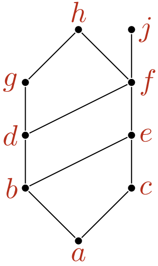
Notable Elements

Definition (Supremum or least upper bound)

An element x is the supremum or the least upper bound of the subset A, denoted by $\bigcup A$, if x is an upper bound that is less than every other upper bound of A.

Example

- $A = \{b, d, g\}$
  Upper bounds: $\{g, h\}$
  $\bigcup A = g$



ω-Complete Partial Orders

Definition (ω-chain)

Let $\mathbf{D} = (D, \sqsubseteq)$ be a poset. A ω-chain of $\mathbf{D}$ is an increasing chain $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$, where $d_i \in D$.

ω-Complete Partial Orders

Definition (ω-complete partial order)

Let $\mathbf{D} = (D, \sqsubseteq)$ be a poset. The poset $\mathbf{D}$ is a ω-complete partial order (ω-cpo) if [Plotkin 1992]

1. There is a least element $\bot \in D$, that is, $\forall x.\ \bot \sqsubseteq x$. The element $\bot$ is called *bottom*.
2. For every ω-chain $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$, the least upper bound $\bigsqcup_{n\in\omega} d_n \in D$ exists.

ω-Complete Partial Orders

Definition (Lifted set)

Let A be a set. The symbol $A_\perp$ denotes the ω-cpo whose elements $A \cup \{\perp\}$ are ordered by $x \sqsubseteq y$, if and only if, $x = \perp$ or $x = y$ [Mitchell 1996]. The ω-cpo $A_\perp$ is called A lifted.

ω-Complete Partial Orders

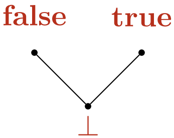
Definition (Lifted set)

Let A be a set. The symbol $A_\perp$ denotes the ω-cpo whose elements $A \cup \{\perp\}$ are ordered by $x \sqsubseteq y$, if and only if, $x = \perp$ or $x = y$ [Mitchell 1996]. The ω-cpo $A_\perp$ is called A lifted.

Examples

The lifted unit type and the lifted Booleans $\mathbf{B}_\perp$.


```
data () = ()
```

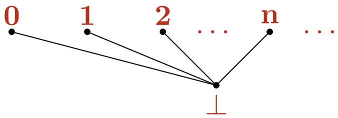



```
data Bool = True | False
```

ω-Complete Partial Orders

Example

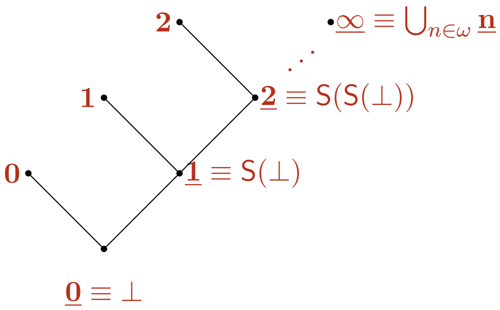
The lifted natural numbers $\mathbf{N}_\perp$.



ω-Complete Partial Orders

Example

The ω-cpo $\mathbf{LN}$ of lazy natural numbers arises from a non-strict successor function, that is, $\mathsf{S}(\perp) \neq \perp$ [Escardó 1993]




```
data Nat = Z
         | S Nat
```

ω-Complete Partial Orders

Definition (Continuous function)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be two ω-cpos. A function $f : D \to D'$ is continuous if [Plotkin 1992]

1. The function is monotone.
2. The function preserves the least upper bounds of the ω-chains, that is,

$$\bigsqcup_{n \in \omega} f(d_n) = f(\bigsqcup_{n \in \omega} d_n),$$

   for all ω-chains $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$.

ω-Complete Partial Orders

Definition (Function space of continuous functions)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be two ω-cpos. The function space of continuous functions is the set [Winskel 1994]

$$[D \to D'] = \{f : D \to D' \mid f \text{ is continous } \}.$$

ω-Complete Partial Orders

Definition (Function space of continuous functions)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be two ω-cpos. The function space of continuous functions is the set [Winskel 1994]

$$[D \to D'] = \{f : D \to D' \mid f \text{ is continous } \}.$$

Theorem

The function space $[D \to D']$ is an ω-cpo.

- $[D \to D']$ can be partially ordered point-wise by

$$f \sqsubseteq g \Leftrightarrow \forall d \in D.\ f(d) \sqsubseteq' g(d).$$

- The bottom element is $\lambda x.\bot_{D'}$.

ω-Complete Partial Orders

Definition

Let $f : D \to D$ be a function, then

$$f^0(d) = d,$$
$$f^{n+1}(d) = f(f^n(d)).$$

Theorem (The Fixed-Point Theorem)

Let $(D, \sqsubseteq)$ be an ω-cpo. Given $f \in [D \to D]$, then

$$\mathrm{Fix}(f) = \bigcup_{n \in \omega} f^n(\bot),$$

is the least fixed-point of f [Winskel 1994], that is,

$$\forall d.\ f(d) \sqsubseteq d \Rightarrow \mathrm{Fix}(f) \sqsubseteq d,$$
$$f(\mathrm{Fix}(f)) = \mathrm{Fix}(f).$$

ω-Complete Partial Orders

Definition (Coalesced sum)

Let $\mathbf{D}_1 = (D_1, \sqsubseteq_1), \ldots, \mathbf{D}_n = (D_n, \sqsubseteq_n)$ be ω-cpos. The coalesced sum, that is, disjoint union with bottom elements identified $\mathbf{D}_1 \oplus \cdots \oplus \mathbf{D}_n$ is the ω-cpo [Plotkin 1992]

$$\left(\bigcup_{i \leq n} \{(i, d) \mid d \in D_i \wedge d \neq \perp\} \right) \cup \perp$$

with the order

$$x \sqsubseteq y \Leftrightarrow x = \perp \text{ or }$$
$$\exists i \leq n. \exists d, d' \in D_i.\ d \sqsubseteq_i d' \wedge x = (i, d) \wedge y = (i, d').$$

ω-Complete Partial Orders

Associated with the coalesced sum are the injection functions

$$in_i : \mathbf{D}_i \to \mathbf{D}_1 \oplus \cdots \oplus \mathbf{D}_n$$
$$in_i(d) = \begin{cases} \bot & \text{if } d = \bot, \\ (i, d) & \text{otherwise}. \end{cases}$$

Domain Model for LTC

Terms: The term language of LTC

From domain theory it is known that a domain model for Terms, where self-application is allowed and where the terms will have values in the Booleans or the lazy natural numbers is a solution to the recursive domain equation [Plotkin 1992]

$$\mathbf{D} \cong \mathbf{B}_{\perp} \oplus \mathbf{LN} \oplus (\mathbf{D} \to \mathbf{D})_{\perp}.$$

Domain Model for LTC

Notation

Let $\mathbf{D}$ be a domain and let ρ be a valuation on $\mathbf{D}$ (a function from the set of variables to $\mathbf{D}$).

- $\rho(x \mapsto \mathbf{d})$: the valuation which maps x to $\mathbf{d}$ and otherwise acts like ρ.
- $\lambda\mathbf{x}.\mathbf{e}$: λ-abstraction on $\mathbf{D}$.

Domain Model for LTC

Convention

$\mathbf{D}$: A solution to the recursive domain equation for LTC.

From terms to functions and viceverse

The domain $\mathbf{D}$ comes equipped with the continuous functions [Barendregt 2004]

$$\mathrm{F} : \mathbf{D} \to [\mathbf{D} \to \mathbf{D}],$$
$$\mathrm{G} : [\mathbf{D} \to \mathbf{D}] \to \mathbf{D}.$$

Domain Model for LTC

Interpretation function

$[\![\ ]\!]_\rho : \text{Terms} \to \mathbf{D}$: (Based on Pitts [1994])

$$
\begin{aligned}
[\![x]\!]_\rho &= \rho(x), \\
[\![\lambda x.t]\!]_\rho &= \mathrm{G}(\lambda \mathbf{d}.[\![t]\!]_{\rho(x \mapsto \mathbf{d})}), \\
[\![t \cdot t']\!]_\rho &= \begin{cases} \mathbf{f}([\![t']\!]_\rho) & \text{if } [\![t]\!]_\rho = \mathrm{G}(\mathbf{f}), \\ \bot & \text{otherwise}, \end{cases} \\
[\![\text{fix } x.t]\!]_\rho &= \mathrm{Fix}(\lambda \mathbf{d}.[\![t]\!]_{\rho(x \mapsto \mathbf{d})}), \\
[\![\text{true}]\!]_\rho &= \mathbf{true},
\end{aligned}
$$

$$
\begin{aligned}
[\![\text{false}]\!]_\rho &= \mathbf{false}, \\
[\![\text{if}]\!]_\rho &= \mathrm{G}(\mathbf{if}), \\
[\![0]\!]_\rho &= \mathbf{0}, \\
[\![\text{succ}]\!]_\rho &= \mathrm{G}(\mathbf{succ}), \\
[\![\text{pred}]\!]_\rho &= \mathrm{G}(\mathbf{pred}), \\
[\![\text{iszero}]\!]_\rho &= \mathrm{G}(\mathbf{iszero}),
\end{aligned}
$$

where

Domain Model for LTC

we omit the use of the injection functions in_i, and the continuous functions $\mathbf{if}$, $\mathbf{succ}$, $\mathbf{pred}$ and $\mathbf{iszero}$ from $\mathbf{D}$ to $\mathbf{D}$ are defined by

$$\mathbf{if}(d) = \begin{cases} \lambda \mathbf{xy.x} & \text{if } d = \mathbf{true}, \\ \lambda \mathbf{xy.y} & \text{if } d = \mathbf{false}, \\ \perp & \text{otherwise}, \end{cases}$$

$$\mathbf{succ}(d) = \begin{cases} \mathbf{n+1} & \text{if } d = \mathbf{n} \in \mathbf{LN}, \\ \underline{\mathbf{n+1}} & \text{if } d = \underline{\mathbf{n}} \in \mathbf{LN}, \\ \perp & \text{otherwise}, \end{cases}$$

Domain Model for LTC

$$\mathbf{pred}(d) = \begin{cases} \mathbf{0} & \text{if } d = \mathbf{0}, \\ d' & \text{if } d = \mathbf{succ}(d'), \\ \bot & \text{otherwise}, \end{cases}$$

$$\mathbf{iszero}(d) = \begin{cases} \mathbf{true} & \text{if } d = \mathbf{0}, \\ \mathbf{false} & \text{if } d = \mathbf{succ}(d'), \\ \bot & \text{otherwise}. \end{cases}$$

Domain Model for LTC

If the LTC equality is interpreted as the equality in $\mathbf{D}$, it is possible verify that the conversion and discrimination rules of LTC are satisfied in $\mathbf{D}$.

Bonus Slides

Monotone Functions

Example (Counter-example of monotone function)

$$\mathbf{halt} : \mathbf{N}_\perp \to \mathbf{B}_\perp$$

$$\mathbf{halt}(n) = \begin{cases} \mathbf{true} & \text{if } n \neq \perp, \\ \mathbf{false} & \text{if } n = \perp. \end{cases}$$

Let $n \in \mathbf{N}_\perp$. Since $\perp \sqsubseteq n$, and not necessarily $\mathbf{halt}(\perp) \sqsubseteq \mathbf{halt}(n)$, that is, $\mathbf{false} \not\sqsubseteq \mathbf{true}$, the $\mathbf{halt}$ function is non-monotone [Schmidt 1986].

Continuous Functions

Example (Monotone but non-continuous function[1])

$$f : [\mathrm{Bool}] \to \mathrm{Bool}$$

$$f(xs) = \begin{cases} \bot & \text{if } xs \text{ is finite}, \\ \mathrm{False} & \text{if } xs = [\mathrm{False}, \mathrm{False}, \ldots], \\ \mathrm{True} & \text{otherwise}. \end{cases}$$

Given $d_0 = []$, $d_1 = [\mathrm{False}]$, $d_2 = [\mathrm{False}, \mathrm{False}]$, $\ldots$, we have

$$\bigsqcup_{n \in \omega} f(d_n) = \bot \neq \mathrm{False} = f([\mathrm{False}, \mathrm{False}, \ldots]) = f\left(\bigsqcup_{n \in \omega} d_n\right),$$

that is, the function f is non-continuous.

[1] http://www.reddit.com/r/types/comments/1ahfh7/intuition_behind_continuity_in_winskels/.

Complete Partial Orders

Definition (Directed set)

Let $\mathbf{D} = (D, \sqsubseteq)$ be a poset. A subset $X \subseteq D$ is directed if $X \neq \emptyset$ and

$$\forall x\ y \in X. \exists z \in X.\ x \sqsubseteq z \wedge y \sqsubseteq z.$$

Definition (Complete partial order)

Let $\mathbf{D} = (D, \sqsubseteq)$ be a poset. The poset $\mathbf{D}$ is a complete partial order (cpo) if [Barendregt 2004]

1. There is a least element $\bot \in D$, that is, $\forall x.\ \bot \sqsubseteq x$. The element $\bot$ is called *bottom*.
2. For every directed $X \subseteq D$, the least upper bound $\bigsqcup X \in D$ exists.

Complete Partial Orders

Note

The Scott domains are built from complete partial orders. See, for example, Gunter and Scott [1990].

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