Verification of Functional Programs
Induction

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Source Code

All the source code have been tested with Agda 2.3.2, Coq 8.4pl3 and Isabelle 2013-2.

The Principle of Mathematical Induction

The principle of mathematical induction

Let $A(x)$ be a propositional function. To prove $A(x)$ for all $x \in \mathbb{N}$, it suffices prove:

- the basis $A(0)$ and
- the induction step, that $A(n) \Rightarrow A(n+1)$, for all $n \in \mathbb{N}$ ($A(n)$ is called the induction hypothesis).

The Principle of Mathematical Induction

First-order logic version

Let $A(x)$ be a formula with free variable x. For each formula $A(x)$:

$$[\,A(0)\ \wedge\ \forall x.A(x) \Rightarrow A(x+1)\,] \Rightarrow \forall x.A(x) \quad \text{(axiom schema of induction)}$$

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Equivalent formulations

$$A(0) \Rightarrow [\,(\forall x.A(x) \Rightarrow A(x+1)) \Rightarrow \forall x.A(x)\,] \qquad \text{(by exportation)}$$

$$A(0) \Rightarrow (\forall x.A(x) \Rightarrow A(x+1)) \Rightarrow \forall x.A(x) \qquad \text{(right-assoc. conditional)}$$

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Inference rule style

$$\frac{A(0) \qquad \forall x.A(x) \Rightarrow A(x+1)}{\forall x.A(x)}$$

Higher-order logic

'The adjetive 'first-order' is used to distinguish the languages... from those in which are predicates having other predicates or functions as arguments, or quantification over functions or predicates, or both.' (Mendelson 1997, p. 56)

The Principle of Mathematical Induction

Higher-order logic

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Second-order logic version

Let X be a predicate variable.

$$\forall X . X(0) \Rightarrow (\forall x . X(x) \Rightarrow X(x+1)) \Rightarrow \forall x . X(x) \qquad \text{(axiom of induction)}$$

The Principle of Mathematical Induction

Historical remark

Dedekind (2005) and Peano (1967) axiom: $1 \in \mathbb{N}$.

The Principle of Mathematical Induction

Remark

Coq generates the induction principles associated to the inductively defined (data) types.

Example (Coq)

The inductive data type for natural numbers.

```
Require Import Unicode.Utf8.

Inductive nat : Set :=
| O : nat
| S : nat → nat.
```

Continued on next slide

The Principle of Mathematical Induction

Example (continuation)

The `Check nat_ind` command yields:

```
nat_ind : ∀ P : nat → Prop,
          P O → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

Example (continuation)

The `Check nat_ind` command yields:

```
nat_ind : ∀ P : nat → Prop,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The `Check nat_rec` command yields:

```
nat_rec : ∀ P : nat → Set,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The Principle of Mathematical Induction

Example (continuation)

The `Check nat_ind` command yields:

```
nat_ind : ∀ P : nat → Prop,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The `Check nat_rec` command yields:

```
nat_rec : ∀ P : nat → Set,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

The `Check nat_rect` command yields:

```
nat_rec : ∀ P : nat → Type,
          P 0 → (∀ n : nat, P n → P (S n)) → ∀ n : nat, P n
```

Implementation remark

What happen if instead of using

```
Inductive nat : Set := 0 : nat | S : nat → nat
```

we renamed the data type nat by

```
Inductive P : Set := 0 : P | S : P → P
```

or we renamed the data constructor S by

```
Inductive nat : Set := 0 : nat | P : nat → nat
```

?

Source: McBride and McKinna (2004)

The Principle of Mathematical Induction

Remark

Isabelle also generates the induction principles associated to the inductively defined (data) types.

Example (Isabelle)

The inductive data type for natural numbers.

```
datatype nat = Z | S nat
```

The Principle of Mathematical Induction

Remark

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Example (Isabelle)

The inductive data type for natural numbers.

```
datatype nat = Z | S nat
```

The `print_theorems` command yields (among others):

```
nat.induct: ?P Z ⇒ ∀x. ?P x ⇒ ?P (S x)) ⇒ ?P ?nat
```

Remark

Agda doesn't generate the induction principles, but the user can use pattern matching on the inductively defined (data) types.

Example (Agda)

The inductive data type for natural numbers.

```
data ℕ : Set where
  zero : ℕ
  succ : ℕ → ℕ
```

Continued on next slide

The Principle of Mathematical Induction

Example (continuation)

The principle of mathematical induction.

```
ℕ-ind : (A : ℕ → Set) →
        A zero →
        (∀ n → A n → A (succ n)) →
        ∀ n → A n
ℕ-ind A A0 h zero     = A0
ℕ-ind A A0 h (succ n) = h n (ℕ-ind A A0 h n)
```

The Principle of Mathematical Induction

Remark

In Agda, Coq and Isabelle, the ‘axiom of induction’ is not an axiom

The Principle of Mathematical Induction

Remark

In Agda, Coq and Isabelle, the 'axiom of induction' is not an axiom (the introduction rules induce the induction principles).

Course-of-Values Induction

Course-of-values induction (strong or complete induction)

Let $A(x)$ be a propositional function. To prove $A(x)$ for all $x \in \mathbb{N}$, it is enough to prove:

$$(\forall 0 \leq k < n)(A(k) \Rightarrow A(n)), \text{ for all } n \in \mathbb{N}.$$

Course-of-Values Induction

Example

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_k + F_{k+1}$, so $F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$.

Course-of-Values Induction

Example

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_k + F_{k+1}$, so $F = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \dots\}$.

Let Φ and $\hat{\Phi}$ be the roots of the equation $x^2 - x - 1$:

$$\Phi = \frac{1+\sqrt{5}}{2} \text{ and } \hat{\Phi} = \frac{1-\sqrt{5}}{2},$$

so $\Phi^2 = \Phi + 1$ and $\hat{\Phi}^2 = \hat{\Phi} + 1$. Then (Bird and Wadler 1988, p. 107.)

$$F_k = \frac{1}{\sqrt{5}}(\Phi^k - \hat{\Phi}^k), \text{ for all } k \in \mathbb{N}.$$

Mathematical and Course-of-Values Induction

Theorem

Mathematical induction and course-of-values induction are equivalent (Winskel 2010).

Structural Induction

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To prove $A(X)$ for all the structures X, it suffices prove (Hopcroft, Motwani and Ullman 2007):

- $A(X)$ for the basis structure(s) of X and

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To prove $A(X)$ for all the structures X, it suffices prove (Hopcroft, Motwani and Ullman 2007):

- $A(X)$ for the basis structure(s) of X and
- given a structure X whose recursive/inductive definition says is formed from $Y_1, \ldots, Y_k$, that $A(X)$ assuming that the properties $A(Y_1), \ldots, A(Y_k)$ hold.

Structural Induction for Lists

Example (Coq)

The parametric inductive data type.

```
Require Import Unicode.Utf8.

Inductive list (A : Type) : Type :=
| nil  : list A
| cons : A → list A → list A.
```

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| nil  : list A
| cons : A → list A → list A.
```

The induction principle.

```
list_ind : ∀ (A : Type) (P : list A → Prop),
           P (nil A) →
           (∀ (a : A) (l : list A), P l → P (cons A a l)) →
           ∀ l : list A, P l
```

Structural Induction for Lists

Example (Isabelle)

The polymorphic inductive data type.

```
datatype 'a list = Nil | Cons 'a "'a list"
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The induction principle.

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list.induct: ?P Nil ⇒ ∀x1 x2. ?P x2 ⇒ ?P (Cons x1 x2)) ⇒
             ?P ?list
```

Structural Induction for Lists

Example (Agda)

The parametric inductive data type.

```
data List (A : Set) : Set where
  []  : List A
  _∷_ : A → List A → List A
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data List (A : Set) : Set where
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The induction principle.

```
List-ind : {A : Set} (B : List A → Set) →
           B [] →
           ((x : A) (xs : List A) → B xs → B (x ∷ xs)) →
           ∀ xs → B xs
List-ind B B[] h []       = B[]
List-ind B B[] h (x ∷ xs) = h x xs (List-ind B B[] h xs)
```

Well-Founded Induction

Definition

Let $\prec$ be a binary relation on a set A. The relation $\prec$ is a **well-founded** relation iff every non-empty subset $S \subseteq A$ has a minimal element, that is,

$$(\forall S \subseteq A)[\, S \neq \emptyset \Rightarrow (\exists m \in S)(\forall s \in S)(s \not\prec m)\,].$$

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$$(\forall S \subseteq A)[\, S \neq \emptyset \Rightarrow (\exists m \in S)(\forall s \in S)(s \not\prec m)\,].$$

Definition (Well-founded induction)

Let $\prec$ be a well-founded relation on a set A and $A(x)$ a propositional function. To prove $A(x)$ for all $a \in A$, it suffices prove:

$$(\forall b \prec a)(A(b) \Rightarrow A(a)), \text{ for all } a \in A.$$

Well-Founded Induction

Example

Let $\prec$ be the well-founded relation on $\mathbb{N}$ given by the graph of the successor function $n \mapsto n+1$.

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Example

Let $\prec$ be the well-founded relation 'less than' on $\mathbb{N}$.

Then course-of-values induction is a special case of well-founded induction.

Well-Founded Induction

Example

'If we take $\prec$ to be the relation between expressions such that $a \prec b$ holds iff a is an immediate sub-expression of b we obtain the principle of structural induction as a special case of well-founded induction.' (Winskel 2010, p. 93)

Empty Type

In type theory $a : A$ denotes that a is a term (or proof term) of type A.

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Under the proposition-as-types principle, the empty type represents the false (absurdity or contradiction) proposition (Sørensen and Urzyczyn 2006).

Therefore `e : EmptyType` represents a contradiction in our formalisation.

Empty Type

Example (Agda)

```
data ⊥ : Set where

⊥-elim : {A : Set} → ⊥ → A
⊥-elim ()  -- The absurd pattern.
```

Empty Type

Example (Coq)

(From the standard library)

```
Inductive Empty_set : Set :=.

Empty_set_rect : ∀ (P : Empty_set → Type) (e : Empty_set), P e
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Inductive Empty_set : Set :=.

Empty_set_rect : ∀ (P : Empty_set → Type) (e : Empty_set), P e

Theorem emptySetElim {A : Set}(e : Empty_set) : A.
  apply (Empty_set_rect (fun _ => A) e).
Qed.
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Empty_set_rect : ∀ (P : Empty_set → Type) (e : Empty_set), P e

Theorem emptySetElim {A : Set}(e : Empty_set) : A.
  apply (Empty_set_rect (fun _ => A) e).
Qed.

Theorem emptySetElim' {A : Set}(e : Empty_set) : A.
  elim e.
Qed.
```

Strictly Positive Inductive Types

Remark

The inductive types can be defined/represented as least fixed-points of appropriated functions (functors).

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Example

Let 1 be the unity type, and $+$ and $\times$ be the operators for disjoint union and Cartesian product, respectively. Then

$$\mathsf{Nat} := \mu X.1 + X,$$
$$\mathsf{List}\ A := \mu X.1 + (A \times X).$$

Strictly Positive Inductive Types

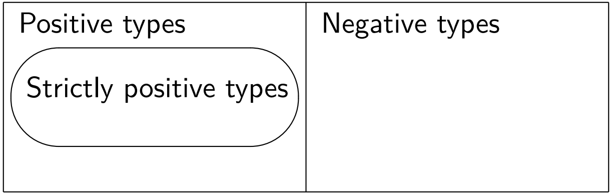
Definition

'The occurrence of a type variable is **positive** iff it occurs within an even number of left hand sides of $\rightarrow$-types, it is **strictly positive** iff it never occurs on the left hand side of a $\rightarrow$-type.' (Abel and Altenkirch 2000, p. 21).

Strictly Positive Inductive Types

Definition

Let $\mu X.F(X)$ be an inductive type. The type $\mu X.F(X)$ is a **strictly positive type** if X occurs strictly positive in $F(X)$.



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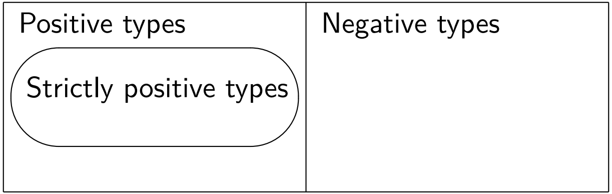

Proof assistants

Agda, Coq and Isabelle accept only strictly positive inductive types.

Strictly Positive Inductive Types

Some issues with non-strictly positive inductive types

- Infinite unfolding
  See source code in the course web page.

Strictly Positive Inductive Types

Some issues with non-strictly positive inductive types

- Infinite unfolding
  See source code in the course web page.
- Proving absurdity
  See source code in the course web page.

Strictly Positive Inductive Types

The following examples of inductive types* are rejected by Agda (Coq and Isabelle) because they are not strictly positive inductive types.

Example (negative type)

$$\mathsf{D} := \mu X . X \to X$$

```
data D : Set where
  lam : (D → D) → D
-- D is not strictly positive, because it occurs to the left
-- of an arrow in the type of the constructor lam in the
-- definition of D.
```

*Adapted from the Coq'Art, Matthes' PhD thesis and Agda's source code.

Strictly Positive Inductive Types

Example (positive, non-strictly positive type)

$$\mathsf{P} := \mu X.(X \to 2) \to 2$$

```
data P : Set where
   p : ((P → Bool) → Bool) → P
-- P is not strictly positive, because it occurs to the left
-- of an arrow in the type of the constructor p in the
-- definition of P.
```

References

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