Ordinals and Typed Lambda Calculus

Set Theory Preliminaries

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Notation

Conventions

- Sets will be denoted by lowercase letters $(a, b, \ldots)$ and uppercase letters $(A, B, \ldots)$.
- Ordinal numbers will be denoted by lowercase Greek letters $(\alpha, \beta, \ldots)$.
- Proper class will be denoted by upper-case sans serif letters $(\mathsf{A}, \mathsf{B}, \ldots)$.

Classes

Definition

A **class** is a collection of sets.

Remark

Every set is a class, but some classes are too large to be sets.

Definition

A class A is a **set** iff $A \subseteq V_\alpha$ (i.e. $A \in V_{\alpha+1}$) for some ordinal number α.

Definition

A **proper class** is a class which is not a set.

Example

The collection of all sets is a proper class.

Classes

Remark

When introducing an axiomatic for set theory based on classes (von Neumann–Bernays–Gödel set theory), Mendelson [Men2015, p. 232-3] wrote:

> *The sets are intended to be those safe, comfortable classes that are used by mathematicians in their daily work, whereas proper classes are thought of as monstrously large collections that, if permitted to be sets (i.e. allowed to belong to other classes), would engender contradictions.*

Some Axiomatic Theories for Set Theory

- Zermelo-Fraenkel set theory (ZF)
- Zermelo-Fraenkel set theory with choice (ZFC)
- von Neumann–Bernays–Gödel set theory (NBG)
- Tarski–Grothendieck set theory (TG)

ZFC's Axioms

Non-rigorous classification

- Axioms stating the existence of sets
- Axioms determining properties of sets
- Axioms for building sets from other sets

ZFC's Axioms Stating the Existence of Sets

Empty (existence) axiom

There is a set having no members:

$$\exists B \, \forall x \, (x \notin B).$$

Infinity axiom

There exists an inductive set:

$$\exists A \, [\, \emptyset \in A \land \forall a \, (a \in A \to \operatorname{succ} a \in A) \,],$$

where

$$\operatorname{succ} a := a \cup \{a\}.$$

ZFC's Axioms Determining Properties of Sets

Extensionality axiom

If two sets have exactly the same members, then they are equal:

$$\forall A \, \forall B \, [\, \forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B \,].$$

Regularity (foundation) axiom

All sets are well-founded:

$$\forall A \, [\, A \neq \emptyset \rightarrow \exists m \, (m \in A \land m \cap A = \emptyset) \,].$$

ZFC's Axioms for Building Sets from other Sets

Pairing axiom

For any sets u and v, there is a set having as members just u and v:

$$\forall a\, \forall b\, \exists C\, \forall x\, (x \in C \leftrightarrow x = a \vee x = b).$$

Union axiom

For any set A, there exists a set B whose elements are exactly the members of the members of A:

$$\forall A\, \exists B\, \forall x\, [\, x \in B \leftrightarrow \exists b\, (b \in A \wedge x \in b)\,].$$

Notation

The set B is denoted by $\bigcup A$. Note that $A \cup B = \bigcup\{A, B\}$.

ZFC's Axioms for Building Sets from other Sets

Power set axiom

For any set a, there is a set whose members are exactly the subsets of a:

$$\forall a \, \exists B \, \forall x \, (x \in B \leftrightarrow x \subseteq a).$$

Subset axiom scheme (axiom scheme of comprehension or separation)

For any propositional function $\varphi(x)$, not containing B, the following is an axiom:*

$$\forall c \, \exists B \, \forall x \, (x \in B \leftrightarrow x \in c \wedge \varphi(x)).$$

Axiom of choice (a version)

For any relation R there is a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$.

*The propositional function $\varphi(x)$ can depend on other variables $t_1, \ldots, t_k$. In this case, we use $\varphi(x, t_1, \ldots, t_k)$ and we universally quantify on variables $t_1, \ldots, t_k$ when using the axiomatic scheme.

ZFC's Axioms for Building Sets from other Sets

Replacement axiom scheme

For any propositional function $\varphi(x, y)$, not containing B, the following is an axiom:*

$$\forall A\,[\forall x(x \in A \to \exists! y\, \varphi(x, y)) \to \exists B\, \forall y(y \in B \leftrightarrow \exists x\,(x \in A \wedge \varphi(x, y)))\,].$$

That is, the replacement axiom scheme asserts that

$$B = \{\, y \mid \exists x(x \in A \wedge \varphi(x, y)\,\} \text{ is a set.}$$

*The propositional function $\varphi(x, y)$ can depend on other variables $t_1, \dots, t_k$. In this case, we use $\varphi(x, y, t_1, \dots, t_k)$ and we universally quantify on variables $t_1, \dots, t_k$ when using the axiomatic scheme.

Order Theory

Remark

Ordering relations can be interchangeably defined from non-strict $(\preceq)$ or strict $(\prec)$ orderings.

Order Theory

Definition

A binary relation $\preceq$ on a set A is a **partial non-strict ordering** iff it satisfies the following properties:

$$\forall x\,(x \preceq x) \qquad \text{(reflexivity)}$$

$$\forall x\,\forall y\,(x \preceq y \preceq x \rightarrow x = y) \qquad \text{(anti-symmetry)}$$

$$\forall x\,\forall y\,\forall z\,(x \preceq y \preceq z \rightarrow x \preceq z) \qquad \text{(transitivity)}$$

The pair $(A, \preceq)$ is a **partially ordered set** (or **poset**).

Order Theory

Definition

A binary relation $\prec$ on a set A is a **partial strict ordering** iff it satisfies the following properties:

$$\forall x\,(\neg(x \prec x)) \qquad \text{(irreflexivity)}$$

$$\forall x\,\forall y\,\forall z\,(x \prec y \prec z \rightarrow x \prec z) \qquad \text{(transitivity)}$$

Order Theory

Theorem

Relationship between partial non-strict orderings and partial strict orderings:*

- Let $\preceq$ be a partial non-strict ordering on a set A. The relation

$$a \prec b := a \preceq b \text{ and } a \neq b$$

  is a partial strict ordering on A.

- Let $\prec$ be a partial strict ordering on a set A. The relation

$$a \preceq b := a \prec b \text{ or } a = b$$

  is a partial non-strict ordering on A.

*See, e.g. [HJ1999, Ch. 2, Theorem 5.6].

Order Theory

Definition

A binary relation $\preceq$ on a set A is a **total non-strict ordering** iff it satisfies the following properties:

$$\forall x\,(x \preceq x) \qquad \text{(reflexivity)}$$

$$\forall x\,\forall y\,(x \preceq y \preceq x \rightarrow x = y) \qquad \text{(anti-symmetry)}$$

$$\forall x\,\forall y\,\forall z\,(x \preceq y \preceq z \rightarrow x \preceq z) \qquad \text{(transitivity)}$$

$$\forall x\,\forall y\,(x \preceq y \vee y \preceq x) \qquad \text{(connexity or totality)}$$

The pair $(A, \preceq)$ is a **totally ordered set**.

Remark

The connexity (totality) property implies reflexivity.

Order Theory

Example

Let's define all the total non-strict ordering on the set $\{1, 2, 3\}$.

(a) (b) (c) (d) (e) (f)

Order Theory

Definition

A binary relation $\prec$ on a set A is a **total strict ordering** iff it satisfies the following properties:

$$\forall x \, (\neg(x \prec x)) \qquad \text{(irreflexivity)}$$
$$\forall x \, \forall y \, \forall z \, (x \prec y \prec z \rightarrow x \prec z) \qquad \text{(transitivity)}$$
$$\forall x \, \forall y \, (x \neq y \rightarrow x \prec y \vee y \prec x) \qquad \text{(connexity or totality)}$$

Order Theory

Definition

A total strict ordering $\prec$ on a set A is a **well-ordering** iff every non-empty subset of A has a least (minimum) element.

Order Theory

Definition

A total strict ordering $\prec$ on a set A is a **well-ordering** iff every non-empty subset of A has a least (minimum) element.

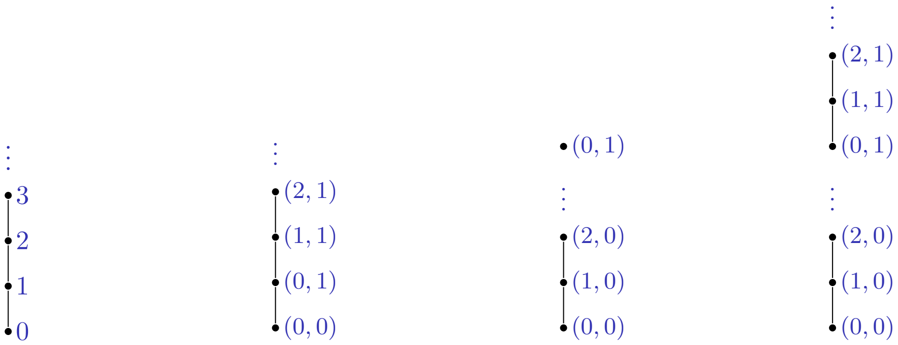
Example

Let's define some well-orderings on the set $\{a, b, c, d\}$. See whiteboard.

Order Theory

Example

Some denumerable well-orderings where $\uplus$ denotes the disjoint union of sets, i.e. $A \uplus B := (A \times \{0\}) \cup (B \times \{1\})$.



(a) ω (b) $\{0\} \uplus \omega$ (c) $\omega \uplus \{0\}$ (d) $\omega \uplus \omega$

References

[HJ1999] Karel Hrbacek and Thomas Jech. Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker, 1999 (1978) (cit. on p. 15).

[Men2015] Elliott Mendelson. Introduction to Mathematical Logic. 6th ed. CRC Press, 2015 (1964) (cit. on p. 4).