Ordinals and Typed Lambda Calculus

Ordinal Notations

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Semester 2018-2

(Last modification: 7th May 2025)

Introduction

Too many numbers to be named

Spector [Spe1955] starts by pointing out that

> *Cantor's second ordinal number class is perhaps the simplest example of a set of mathematical objects which cannot all be named in one language.*

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Remark

Recall that a language is a subset of words over an alphabet.

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Yes! We can name any natural number by a word over the alphabet $\{0, 1, 2, \ldots, 9\}$.

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Can all natural numbers be named in one language?

Yes! We can name any natural number by a word over the alphabet $\{0, 1, 2, \ldots, 9\}$.

That was easy because the set of natural numbers is denumerable.

Introduction

Question

Can all real numbers be named in one language?

Notation Systems for the Ordinals below ϵ_0

Theorem

Every ordinal α less than ϵ_0 has a normal form

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_n},$$

where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ are ordinals and $\beta_i < \alpha$ [Poh2009, p. 33].

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Definition

From the above theorem, we can name any ordinal less than ϵ_0 by a word over the alphabet $\{+, 0, \omega^{\cdot}\}$. By coding these words in natural numbers, we get a **notation system for the ordinals below $\boldsymbol{\epsilon_0}$** [Poh2009, p. 33].

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Remark

The ordinal ϵ_0 is the smallest ordinal that has no a name in terms of ω.

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Representation using trees

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There is a one-to-one correspondence between finite rooted trees and ordinals below ϵ_0 given by [Der1993]:

i) The one-node tree represents the ordinal 0.

ii) The tree with sub-trees representing the ordinals $\alpha_1, \ldots, \alpha_n$ represents the ordinal $\omega^{\alpha_1} \# \cdots \# \omega^{\alpha_n}$.

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Kleene's $\mathcal{O}$

Definition

We inductively define the **notation system** $\mathcal{O}$ and the **well-founded ordering** $<_{\mathcal{O}}$ [Coo2004, Definition 16.2.29, p. 358]:

i) *We start by giving the ordinal* 0 *notation* 1. *Assume all ordinals less than* α *have been assigned notations, and* $<_{\mathcal{O}}$ *has been defined on these notations.*

ii) *Say* $\alpha = \beta + 1$, *and* β *has notation* x.

   Then α *gets notation* 2^x *and we add* $\langle z, 2^x \rangle$ *to* $<_{\mathcal{O}}$ *for each* z *such that* $z = x$ *or* $z <_{\mathcal{O}} x$ *already.*

iii) *Say* α *is a limit ordinal, and* $\langle \varphi_e(n) : n \in \mathbb{N} \rangle$ *is a list of notations for ordinals with limit* α, *and* $\forall n [\varphi_e(n) <_{\mathcal{O}} \varphi_e(n+1)]$ *already.*

   Then give α *notation* $3 \cdot 5^e$, *and add* $\langle z, 3 \cdot 5^e \rangle$ *to* $<_{\mathcal{O}}$ *for all* z *for which* $z <_{\mathcal{O}} \varphi_e(n)$ *already, some* $n \geq 0$.

Kleene's $\mathcal{O}$

Remark

We could use the notations zero, $\mathrm{succ}(x)$ and $\lim(e)$ instead of the notations $1, 2^x$ and $3 \cdot 5^e$.*

*See, e.g. [Frá2017].

Constructive Ordinals and Computable Ordinals

Definition

The **constructive ordinals** (second definition) are the ordinals notated by $\mathcal{O}$ [Coo2004, Definition 16.2.29, p. 358].

*See, e.g. [Coo2004, Definition 16.2.25, p. 358], [Rog1992, p. 211] and [AK2000, p. 61].

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Definition

A countable ordinal is **computable** iff it is finite or it is isomorphic to a computable well-ordering $(A, \prec)$.*

*See, e.g. [Coo2004, Definition 16.2.25, p. 358], [Rog1992, p. 211] and [AK2000, p. 61].

Constructive Ordinals and Computable Ordinals

Theorem

An ordinal α is constructive iff α is a computable ordinal.*

*See, e.g. [Rog1992, Corollary XIX and Theorem XX, p. 211] and [AK2000, § 4.7, p. 62].

References

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