Ordinals and Typed Lambda Calculus Defining Ordinals in Martin-Löf Type Theory

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# Hilbert Ordinals

### Definition

Hilbert [Hil1967] (originally published as [Hil1926]) axiomatically defined the ordinals of the (cumulative) second number class. Let  $\rm N$  and  $\rm O$  be unary predicates representing natural and ordinal numbers, respectively.

```
N zero,

\forall n [Nn \to N(\operatorname{succ}(n))],

\{P \operatorname{zero} \land \forall n [Pn \to P(\operatorname{succ}(n)]\} \to \forall n (Nn \to Pn),

O zero,

\forall n [On \to O(\operatorname{succ}(n))],

\forall n [Nn \to O(f(n))] \to \forall n [O(\lim f(n))],

\{P \operatorname{zero} \land \forall n [Pn \to P(\operatorname{succ}(n))] \land \forall n [P(f(n)) \to P(\lim f(n)]\} \to \forall n (On \to Pn).
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## Hilbert Ordinals

Discussion

Can we define a data type for representing ordinal numbers from Hilbert's axioms?

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#### Remark

Cumulative second number class have been studied by various authors (see, e.g. [CK1937; How1972; Ste1972; Mar2001; CHS1997; Mar1984]).

## Martin-Löf's Ordinals

Remark

Martin-Löf defined ordinals in his type theory [Mar1984, p. 83]. He did not mention Hilbert but Cantor when given his definition.

Introduction rules

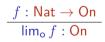
The introduction rules for Martin-Löf's ordinals are the following ones.

zero : Nat

 $\frac{n:\mathsf{Nat}}{\mathsf{succ}\,n:\mathsf{Nat}}$ 

 $zero_o : On$ 

 $\frac{n:\mathsf{On}}{\mathsf{succ}_{\mathsf{o}}\,n:\mathsf{On}}$ 



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