Ordinals and Typed Lambda Calculus

Defining Ordinals in Martin-Löf Type Theory

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Hilbert Ordinals

Definition

Hilbert [Hil1967] (originally published as [Hil1926]) axiomatically defined the ordinals of the (cumulative) second number class. Let N and O be unary predicates representing natural and ordinal numbers, respectively.

$$\mathrm{N}\,\mathrm{zero},$$
$$\forall n\,[\mathrm{N}n \to \mathrm{N}(\mathrm{succ}(n))],$$
$$\{P\,\mathrm{zero} \wedge \forall n\,[Pn \to P(\mathrm{succ}(n)]\} \to \forall n\,(\mathrm{N}n \to Pn),$$

$$\mathrm{O}\,\mathrm{zero},$$
$$\forall n\,[\mathrm{O}n \to \mathrm{O}(\mathrm{succ}(n))],$$
$$\forall n\,[\mathrm{N}n \to \mathrm{O}(f(n))] \to \forall n\,[\mathrm{O}(\lim f(n))],$$
$$\{P\,\mathrm{zero} \wedge \forall n\,[Pn \to P(\mathrm{succ}(n))] \wedge \forall n\,[P(f(n)) \to P(\lim f(n)]\} \to \forall n\,(\mathrm{O}n \to Pn).$$

Hilbert Ordinals

Discussion

Can we define a data type for representing ordinal numbers from Hilbert's axioms?

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Remark

Cumulative second number class have been studied by various authors (see, e.g. [CK1937; How1972; Ste1972; Mar2001; CHS1997; Mar1984]).

Martin-Löf's Ordinals

Remark

Martin-Löf defined ordinals in his type theory [Mar1984, p. 83]. He did not mention Hilbert but Cantor when given his definition.

Introduction rules

The introduction rules for Martin-Löf's ordinals are the following ones.

$$\frac{}{\mathsf{zero} : \mathsf{Nat}} \qquad \frac{n : \mathsf{Nat}}{\mathsf{succ}\, n : \mathsf{Nat}}$$

$$\frac{}{\mathsf{zero}_{\mathsf{o}} : \mathsf{On}} \qquad \frac{n : \mathsf{On}}{\mathsf{succ}_{\mathsf{o}}\, n : \mathsf{On}} \qquad \frac{f : \mathsf{Nat} \to \mathsf{On}}{\mathsf{lim}_{\mathsf{o}}\, f : \mathsf{On}}$$

References

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