Ordinals and Typed Lambda Calculus

Definable Ordinals in the Lambda Calculus

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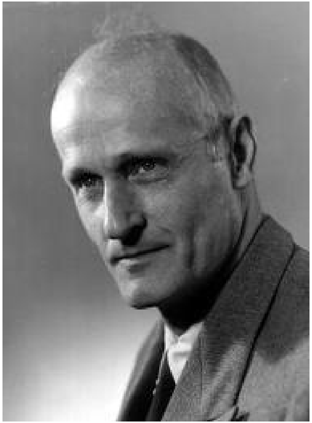
Introduction

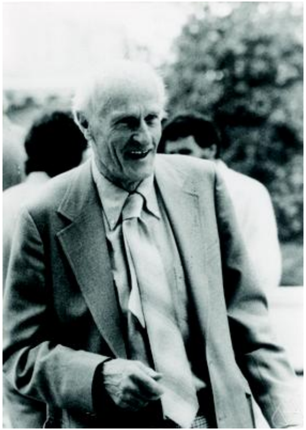
Alonzo Church (1903 – 1995)*

*Figures sources: History of computers, Wikipedia and MacTutor History of Mathematics.

Introduction

Stephen Cole Kleene (1909 – 1994)*





*Figures sources: MacTutor History of Mathematics and Oberwolfach.

Introduction

Some remarks

- Church and Kleene defined the λ-definable ordinals in [Kle1937], [CK1937], [Chu1938] and [Kle1938].
- The λ-definable ordinals are a proper subset of the set of countable ordinals.

Starting the Representation

Basic combinators

The representation of countable ordinals has the following basic combinators:

$$
\begin{aligned}
0_{\mathsf{o}} &:= \lambda m.\, m\, \mathsf{c}_1, \\
\mathsf{succ}_{\mathsf{o}} &:= \lambda\, a\, m.\, m\, \mathsf{c}_2\, a, \\
\mathsf{lim}_{\mathsf{o}} &:= \lambda\, a\, r\, m.\, m\, \mathsf{c}_3\, a\, r.
\end{aligned}
$$

Ordinals Representation

Representation

The **representation of countable ordinals** in the λ-calculus is inductively defined by [CK1937]:

1. If a combinator a represents an ordinal α, and $\mathsf{a}' =_\beta \mathsf{a}$, then a' also represents α.
2. The combinator $\mathsf{0_o}$ represents the ordinal 0.
3. If a combinator a represents an ordinal α, then $\mathsf{succ_o}\,\mathsf{a}$ represents the successor of α.
4. If the ordinal α is the limit of an increasing ω-sequence of ordinals $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ and if r is a combinator such that the λ-terms

   $$\mathsf{r}\,\mathsf{0_o}, \mathsf{r}\,(\mathsf{succ_o}\,\mathsf{0_o}), \mathsf{r}\,(\mathsf{succ_o}\,(\mathsf{succ_o}\,\mathsf{0_o})), \ldots$$

   represent the ordinals $\alpha_0, \alpha_1, \alpha_2, \ldots$, respectively, then $\mathsf{lim_o}\,\mathsf{0_o}\,\mathsf{r}$ represents α.

Ordinals Representation

Example

The finite ordinals are representable by

$$\begin{aligned} 0_{\mathsf{o}} &:= \lambda m.\, m\, \mathsf{c}_1, \\ (\mathsf{n}+1)_{\mathsf{o}} &:= \mathsf{succ}_{\mathsf{o}}\, \mathsf{n}_{\mathsf{o}}. \end{aligned}$$

Hence,

$$\begin{aligned} 0_{\mathsf{o}} &=_\beta \lambda m.\, m\, \mathsf{c}_1, \\ 1_{\mathsf{o}} &=_\beta \lambda m.\, m\, \mathsf{c}_2\, (\lambda m.\, m\, \mathsf{c}_1), \\ 2_{\mathsf{o}} &=_\beta \lambda m.\, m\, \mathsf{c}_2\, (\lambda m.\, m\, \mathsf{c}_2\, (\lambda m.\, m\, \mathsf{c}_1)), \\ 3_{\mathsf{o}} &=_\beta \lambda m.\, m\, \mathsf{c}_2\, (\lambda m.\, m\, \mathsf{c}_2\, (\lambda m.\, m\, \mathsf{c}_2\, (\lambda m.\, m\, \mathsf{c}_1))). \end{aligned}$$

Ordinals Representation

Example

Recall that $\mathsf{I} := \lambda x.x$. The first transfinite countable ordinal ω is representable by

$$\omega_{\mathsf{o}} := \mathsf{lim}_{\mathsf{o}}\, 0_{\mathsf{o}}\, \mathsf{I}.$$

Ordinals Representation

Example

Since that $\omega \cdot 2$ can be defined by

$$\omega \cdot 2 = \lim_{n \in \mathbb{N}} \langle \omega, \omega + 1, \omega + 2, \ldots \rangle,$$

to represent this ordinal, we need to define a combinator r such that

$$\begin{aligned} \mathsf{r}\,\mathsf{0_o} &=_\beta \omega_\mathsf{o}, \\ \mathsf{r}\,(\mathsf{n}+\mathsf{1})_\mathsf{o} &=_\beta \mathsf{succ_o}\,(\mathsf{r}\,\mathsf{n_o}). \end{aligned}$$

Lambda Definable Ordinals

Definition

A (countable) ordinal is $\boldsymbol{\lambda}$-**definable** iff there is a combinator representing it [CK1937, p. 14].

Example

The finite ordinals and the ordinals ω and $\omega \cdot 2$ are λ-definable.

Lambda Definable Ordinals

Theorem

There are countable ordinals which are not λ-definable [CK1937, p. 14].

Proof.

The λ-terms are denumerable but the countable ordinals are not. Therefore, there is a least countable ordinal ξ which is not λ-definable. Moreover, any countable ordinal greater than ξ is neither λ-definable. ∎

Lambda Definable Ordinals

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Remark

Note that the above proof is not constructive.

Lambda Definable Ordinals

Remark

The least countable ordinal which is not λ-definable is denoted ω_1^{CK}, the Church-Kleene ω_1.*

*See, e.g. [Mos2009].

Constructive Ordinals

Definition

An ordinal α is **constructive** (first definition) iff α is λ-definable [Chu1938].

Lambda Definable Ordinal Functions

Example

Addition, multiplication and exponentiation on λ-definable ordinals are λ-definable functions.

Lambda Definable Ordinal Functions

Example

It is possible to define a predecessor function on λ-definable ordinals with the following behaviour:

$$\begin{aligned} \mathsf{pred_o}\ \mathsf{0_o} &=_\beta \mathsf{0_o}, \\ \mathsf{pred_o}\ (\mathsf{succ_o}\ \mathsf{n_o}) &=_\beta \mathsf{n_o} \\ \mathsf{pred_o}\ (\mathsf{lim_o}\ \mathsf{n_o}\ \mathsf{r}) &=_\beta \mathsf{lim_o}\ \mathsf{n_o}\ \mathsf{r} \end{aligned}$$

In relation to the third equation, Church and Kleene [CK1937, Footnote 9] wrote that it was somewhat arbitrary.

Lambda Definable Ordinal Functions

Theorem

Let CO be the set of countable ordinals. The following function is not λ-definable [Chu1938]:

$$\varphi : \mathrm{CO} \times \mathrm{CO} \to \mathrm{CO}$$

$$\varphi(\alpha, \beta) := \begin{cases} 0, & \text{if } \alpha < \beta; \\ 1, & \text{if } \alpha = \beta; \\ 2, & \text{if } \alpha > \beta. \end{cases}$$

Lambda Definable Ordinal Functions

Remark

In relation to the previous theorem, Church wrote:

> *This is not surprising. It is, for instance, not difficult to give examples of pairs of constructive definitions of ordinals such that the question whether the ordinals defined are equal, or which of the two is greater, depends on this or that unsolved problem of number theory; and indeed this may be done without employing any ordinal greater than ω^2. [Chu1938, p. 231]*

References

[Chu1938] Alonzo Church. The Constructive Second Number Class. Bulletin of the American Mathematical Society 44.4 (1938), pp. 224–232. DOI: 10.1090/S0002-9904-1938-06720-1 (cit. on pp. 4, 14, 17, 18).

[CK1937] Alonzo Church and S. C. Kleene. Formal Definitions in the Theory of Ordinal Numbers. Fundamenta Mathematicae 28.1 (1937), pp. 11–21. URL: http://matwbn.icm.edu.pl/ksiazki/fm/fm28/fm2813.pdf (cit. on pp. 4, 6, 10–12, 16).

[Kle1937] S. C. Kleene. On Notation for Ordinal Numbers. Preliminary Report. Bulletin of the American Mathematical Society 41.1 (1937), p. 41. DOI: 10.1090/S0002-9904-1937-06484-6 (cit. on p. 4).

[Kle1938] S. C. Kleene. On Notation for Ordinal Numbers. The Journal of Symbolic Logic 3.4 (1938), pp. 150–157. DOI: 10.2307/2267778 (cit. on p. 4).

[Mos2009] Yiannis N. Moschovakis. Descriptive Set Theory. 2nd ed. American Mathematical Society, 2009 (1980) (cit. on p. 13).