Ordinals and Typed Lambda Calculus

Countable and Uncountable Ordinals

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Cantor's First and Second Number Classes

Cantor's first number class are the finite ordinals and his second number class are the denumerable ordinals. In words of Ivorra Castillo [Ivo2013, p. 293]:

> *Según explicaba [Cantor], los números transfinitos se obtienen mediante dos principios. El 'primer principio de generación' consiste en añadir una unidad. Es el principio que, por sí sólo, genera los números naturales:* $0, 1, 2, 3, \ldots$ *A éstos los llamó 'números transfinitos de primera especie'.*

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>
> *Ahora bien, Cantor afirmaba que, cuando tenemos una sucesión inacabada de números transfinitos, siempre podemos postular la existencia de un nuevo número transfinito como inmediato posterior a todos ellos, y a esto lo llamó el 'segundo principio de generación'.*

(continued on next slide)

Cantor's First and Second Number Classes

(continuation)

> *Así, tras la sucesión de todos los números de primera especie, el segundo principio nos da la existencia de un nuevo número transfinito, el primero de los números de segunda especie, al que Cantor llamó ω. A éste podemos aplicarle de nuevo el primer principio, para obtener $\omega + 1$, $\omega + 2$, etc . . .*

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>
> *Cantor definió los números transfinitos de segunda especie como los números transfinitos que dejan tras de sí una cantidad numerable de números transfinitos.*

Church's Redefinition of the Second Number Class

Note that Cantor's first and second number classes are disjoint. Church redefined the second number class by including the first number class on it [Chu1938, p. 225]:

> *The second number class may be described as the simply ordered set which results when we take 0 as the first (or least) element of the set and allow the two following processes of generation: (1) given any element of the set, to generate the element which next follows it (the least element greater than it); (2) given any infinite increasing sequence of elements, of the order type of the natural numbers, to generate the element which next follows the sequence (the least element greater than every element of the sequence). The elements of the set are ordinals.*

Number Classes Terminology

Remark

In relation to the current use of the number classes terminology, Hancock [Han2008, p. 10] wrote:

> *This terminology [first and second number classes] comes from Cantor. You'll probably encounter it. But beware, sometimes people mean slightly different things by this 'number class' talk. Nowadays, most people probably understand number-classes cumulatively, so that the second number class contains the first number class. Whereas for Cantor himself, the number classes were disjoint.*

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Question

Do you want to know some countable ordinals? The fun can start in Baez's (three parts) blog 'Large Countable Ordinals'.*

*Available at
`https://johncarlosbaez.wordpress.com/2016/06/29/large-countable-ordinals-part-1/` .

The First Epsilon Ordinal

A description of ϵ_0

The ordinal ϵ_0 is defined by

$$\epsilon_0 := \sup \left\{ \omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \omega^{\omega^{\omega^{\omega}}}, \ldots \right\}.$$

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Note that ϵ_0 is a (the least) fixed-point of the exponential function $\lambda x.\omega^x$, that is

$$\epsilon_0 = \omega^{\epsilon_0}.$$

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Is ϵ_0 a countable ordinal?

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Question

Is ϵ_0 a countable ordinal? Yes!

Fundamental Sequences

Definition

A **ω-sequence** is an infinite sequence of the order-type of the natural numbers.

Definition

Let α be a limit countable ordinal and let $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ be an increasing ω-sequence of ordinals such that

$$\alpha = \sup \{ \alpha_i \mid \alpha_i \in \langle \alpha_n \rangle_{n \in \mathbb{N}} \}.$$

The increasing ω-sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a **fundamental sequence** for the ordinal α.*

*See, e.g. [Rog1992] and [Rat2006]. Some authors require that the ω-sequence be strictly increasing. Other authors allow non-decreasing ω-sequences as fundamental sequences for successor ordinals.

Fundamental Sequences

Notation

Given a fundamental sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ for α, we define

$$\lim_{n \in \mathbb{N}} \alpha_n := \sup \{ \alpha_i \mid \alpha_i \in \langle \alpha_n \rangle_{n \in \mathbb{N}} \}.$$

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Example

Some fundamental sequences.

$$\omega = \lim_{n\in\mathbb{N}} \langle 0, 1, 2, \ldots \rangle,$$

$$\omega \cdot 2 = \lim_{n\in\mathbb{N}} \langle \omega, \omega + 1, \omega + 2, \ldots \rangle,$$

$$\omega^2 = \lim_{n\in\mathbb{N}} \langle \omega, \omega \cdot 2, \omega \cdot 3, \ldots \rangle,$$

$$\omega^\omega = \lim_{n\in\mathbb{N}} \langle \omega, \omega^2, \omega^3, \ldots \rangle,$$

$$\epsilon_0 = \lim_{n\in\mathbb{N}} \langle \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \rangle.$$

Fundamental Sequences

Remark

When working with countable ordinals it is common to use fundamental sequences instead of the actual ordinals.

The Countable Ordinals are Non-Denumerable

Theorem

The collection of all countable ordinals is a set.

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Proof*

1. We define the following propositional functions:

$$\mathrm{wo}(x) := x \text{ is a well-ordered set},$$
$$\mathrm{ot}(x, y) := y \text{ is the order-type of } x,$$
$$\varphi(x, y) := [\,\mathrm{wo}(x) \wedge \mathrm{ot}(x, y)] \vee [\neg\mathrm{wo}(x) \wedge y = 0\,].$$

2. Using the replacement axiom scheme on $\mathcal{P}(\omega \times \omega)$ and $\varphi(x, y)$ we know that

$$S = \{\, y \mid \exists x\, (x \in \mathcal{P}(\omega \times \omega) \wedge \varphi(x, y)\,\} \text{ is a set}.$$

*By Juan Carlos Agudelo-Agudelo, personal communication.

The Countable Ordinals are Non-Denumerable

Proof (continuation).

3. Since any denumerable ordinal is isomorphic to some well-ordering on ω (or to some subset of ω if the ordinal is finite), then any countable ordinal belongs to the set S. Hence, the collection of the countable ordinals is a set. ∎

The Countable Ordinals are Non-Denumerable

Proof (continuation).

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Question

Is the previous proof a constructive proof?

The Countable Ordinals are Non-Denumerable

Theorem

The set of all ordinals in Cantor's second class number is non-denumerable.*

Question

Can you think in an one-to-one correspondence between the set of Cantor's second class number and the real numbers?

Theorem

The set of the countable ordinals is non-denumerable.

*See, e.g. [Sie1965, Theorem 2, p. 370].

Uncountable Ordinals

Definition

An **uncountable ordinal** is an ordinal whose cardinality is uncountable.

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Example

The first uncountable ordinal, denoted by ω_1, is the supremum of the set of the countable ordinals.

Cantor's n-th Number Class

Definition

Cantor's first number class are the finite ordinals, his second number class are the ordinals of cardinal $\aleph_0$, his third number class are the ordinals of cardinal $\aleph_1$, and so on.*

*See, e.g. [Rus1938, § 290].

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Cantor's first number class are the finite ordinals, his second number class are the ordinals of cardinal $\aleph_0$, his third number class are the ordinals of cardinal $\aleph_1$, and so on.*

Example

The first uncountable ordinal ω_1 is a 3rd number class.

*See, e.g. [Rus1938, § 290].

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