

MT8002 Elements of Set Theory
Chapter 8. Ordinal Numbers
Section 8.2. Ordinal Numbers

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

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Ordinal Numbers

Basic Properties of Ordinals

The Burali-Forti Paradox

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Generalisation of natural numbers via two abstractions

Cantor's definitions of ordinal and cardinal numbers were "definitions by abstraction" (Ivorra Castillo 2013, pp. 293-4):

"Aquí [Cantor ([1895–1897] 1955)] define Cantor el 'ordinal' de un conjunto ordenado como el concepto al que llegamos cuando hacemos abstracción de la naturaleza de sus elementos y conservamos únicamente su ordenación, de modo que dos conjuntos tienen el mismo ordinal si y sólo si sus elementos pueden ponerse en correspondencia biunívoca conservando el orden. Por otra parte, el 'cardinal' de un conjunto es el concepto al que llegamos cuando hacemos abstracción de la naturaleza de sus elementos, así como de toda posible ordenación."

Introduction

A brief history of the ordinal numbers definition

Cantor (1897)	Order-types of well-ordered sets.
Russell (1903)	Equivalence classes of well-ordered sets.
Mirimanoff (1917)	Every well-ordered set looks like a set well-ordered by inclusion or membership.
von Neumann (1923)	Canonical well-ordered sets.
von Neumann (1928)	Transitive sets well-ordered by the membership relation.
Robinson (1937)	Sets satisfying some properties without using the theory of ordered sets.

Introduction

Definition

Let $(X, <)$ be a well-ordered set. The (Cantor) **order-type** of $(X, <)$ is the **collection** of all well-ordered sets order-isomorphic to it.

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[‡]See, e.g., (van Heijenoort 1967, p. 346) and (Moore 1982, p. 52).

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Problem

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Solution

Ordinal numbers were defined by von Neumann, based on the work of Mirimanoff, without any need for axioms beyond those in ZF.

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Some ideas for defining the ordinals numbers

- To pick from each **class** of order-isomorphic well-ordered sets a **unique set** to represent them all.

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- Since well-ordered sets are strong comparable (either they are order-isomorphic or one is order-isomorphic to an initial segment of the other, Theorem 7.5) the representatives will be linearly ordered by the relation “is order-isomorphic to an initial segment of”.

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- To pick from each **class** of order-isomorphic well-ordered sets a **unique set** to represent them all.
- Since well-ordered sets are strong comparable (either they are order-isomorphic or one is order-isomorphic to an initial segment of the other, Theorem 7.5) the representatives will be linearly ordered by the relation “is order-isomorphic to an initial segment of”.
- *“It would be even nicer if the ‘order-isomorphism to an initial segment’ took some very simple form—simplest of all would surely be the identity function, which would entail that given any two such representative sets, one would be a subset of the other.”*
(p. 203)

Introduction

Some well-ordered sets we know something about

- The set ω is the canonical well-ordered set, so we might choose it as a representative.
- Since every **finite** well-ordered set is order-isomorphic to some $n \in \omega$, we might also choose them as representatives.
- **No duplication so far:** “*distinct natural numbers are not order-isomorphic and no natural number is order-isomorphic to ω .*” (p. 203)
- **No gap so far:** “*every finite well-ordered set is represented by a member of ω , and [...] any well-ordered set that isn't finite, i.e. is infinite, is, or contains an initial segment, order-isomorphic to ω , so cannot fit 'below' ω .*” (p. 203-204)

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Which aspects of ω and its members shall we exploit?

For all (sets) members of ω and ω :

- (a) the well-ordering relation $<$ is the membership relation $\in$ which is one of the fundamental relations of ZF;

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For all (sets) members of ω and ω :

- (a) the well-ordering relation $<$ is the membership relation $\in$ which is one of the fundamental relations of ZF;
- (b) any member of the set is also a subset of it.

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Historical remark

“You’ve been set up! Historically this representation of $\mathbb{N}$ within ZF stemmed from von Neumann’s representation of the ordinals, which is what we’re working up to.” (p. 203)

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Ordinal Numbers

Definition (p. 204)

A set α is an **ordinal** if

- (a) the set α is well-ordered by the relation $\in$, and
- (b) if $\beta \in \alpha$ then $\beta \subseteq \alpha$ (i.e. the set α is $\in$ -transitive (or transitive)).

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The property (b) can be replaced by the equivalent property

- (b') if $\gamma \in \beta$ and $\beta \in \alpha$ then $\gamma \in \alpha$.

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Notation

We shall use lowercase **Greek** letters for representing ordinals.

Ordinal Numbers

Example

The set $\mathbf{3} \doteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ is an ordinal because

- (a) the set $\mathbf{3}$ is well-ordered by the relation $\in$, and
- (b) $\mathbf{0} \subseteq \mathbf{3}$, $\mathbf{1} \subseteq \mathbf{3}$ and $\mathbf{2} \subseteq \mathbf{3}$.

Ordinal Numbers

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Example

All $\mathbf{n} \in \omega$ and ω itself are ordinals.

Ordinal Numbers

Question

Are there any other ordinals?

Ordinal Numbers

Technical remark

In the following exercises “*use the definition of ordinal and properties of well-ordered sets, **rather than** the Axiom of Foundation or any of its consequences*”.

“*Our preferred arguments essentially use only **well-ordering**, which is intrinsic to the whole point of ordinals. The whole theory of ordinals can be developed entirely **without** the Axiom of Foundation.*” (p. 205)

Ordinal Numbers

Exercise 8.3

- (a) Let $n \in \omega$. Show that $\omega \notin n$ and that $\omega \notin \omega$.
- (b) Show that ω^+ , i.e. $\omega \cup \{\omega\}$, is an ordinal.
- (c) Is ω^+ equal to ω or some $n \in \omega$?

Ordinal Numbers

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- (c) Is ω^+ equal to ω or some $n \in \omega$?

Solution

In the whiteboard.

Ordinal Numbers

Exercise

We know that $\omega + 1 \not\cong n$, for all $n \in \omega$. We also know that $\omega \not\cong \omega + 1$.

Show that ω^+ is order-isomorphic to $\omega + 1$.

Ordinal Numbers

Exercise

We know that $\omega + 1 \not\cong n$, for all $n \in \omega$. We also know that $\omega \not\cong \omega + 1$.

Show that ω^+ is order-isomorphic to $\omega + 1$.

Solution

In the whiteboard.

Ordinal Numbers

Remark

“So ω^+ is an ordinal we hadn't listed before. We shall show that there are further ordinals, enough to represent all well-ordered sets.” (p. 206)

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Exercise 8.4

Let α be an ordinal and let $\beta \in \alpha$.

- (a) Show that β is an ordinal.
- (b) Show that $\beta \doteq \text{Seg}_\alpha(\beta)$.

Basic Properties of Ordinals

Exercise 8.4

Let α be an ordinal and let $\beta \in \alpha$.

- (a) Show that β is an ordinal.
- (b) Show that $\beta \doteq \text{Seg}_\alpha(\beta)$.

Remark

So that any member of an ordinal is **also an ordinal**; and an initial segment of an ordinal **is an ordinal**.

Basic Properties of Ordinals

Theorem 7.6 (Principle of transfinite induction)

Let X be a well-ordered set. For all subsets $A \subseteq X$ such that:

- (a) **Base step:** the least element of X is in A .
- (b) **Inductive step:** for all other $x \in X$, if $\text{Seg}_X(x) \subseteq A$ then $x \in A$.

Then $A \doteq X$.

Basic Properties of Ordinals

Theorem 7.6 (Principle of transfinite induction)

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- (a) **Base step:** the least element of X is in A .
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Then $A \doteq X$.

Theorem 8.1 (Principle of transfinite induction for ordinals)

Let α be an ordinal. For all subsets $A \subseteq \alpha$ such that:

- (a) **Base step:** $0 \in A$.
- (b) **Inductive step:** for all [other] $\beta \in \alpha$, if $\beta \subseteq A$ then $\beta \in A$.

Then $A \doteq \alpha$.

Basic Properties of Ordinals

Technical remark

Note that we can exclude the least element 0 from the inductive step of Theorem 8.1 because it trivially satisfies the condition.

Basic Properties of Ordinals

Remark

Note that Theorem 8.1 is a **rephrasing** of Theorem 7.6 for the case in which the well-ordered set X is an ordinal α , using the results of Exercise 8.4:

- (a) the least element of α is the ordinal 0 and
- (b) the initial segment $\text{Seg}_\alpha(\beta)$ is the ordinal β itself.

Basic Properties of Ordinals

The following “*theorem tells us that if, as we hope, each class of order-isomorphic well-ordered sets contains an ordinal, then this ordinal is **unique**.*” (p. 205)

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Theorem 8.2

Suppose that α and β are order-isomorphic ordinals. Then $\alpha \doteq \beta$.

Basic Properties of Ordinals

The following “*theorem tells us that if, as we hope, each class of order-isomorphic well-ordered sets contains an ordinal, then this ordinal is **unique**.*” (p. 205)

Theorem 8.2

Suppose that α and β are order-isomorphic ordinals. Then $\alpha \doteq \beta$.

Proof

In the whiteboard.

Basic Properties of Ordinals

Theorem 8.3

The **class** of ordinals is **well-ordered** by the relation $\in$, in the following sense:

- (a) for all ordinals α , $\alpha \notin \alpha$;
- (b) for all ordinals α , β and γ , if $\alpha \in \beta$ and $\beta \in \gamma$ then $\alpha \in \gamma$;
- (c) for all ordinals α and β , $\alpha \in \beta$ or $\alpha \doteq \beta$ or $\beta \in \alpha$;
- (d) any non-empty **set** X of ordinals contains a least element.

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- (c) for all ordinals α and β , $\alpha \in \beta$ or $\alpha \doteq \beta$ or $\beta \in \alpha$;
- (d) any non-empty **set** X of ordinals contains a least element.

Proof

In the whiteboard.

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Exercise 8.7

Suppose that there is a set $\mathcal{O}$ whose members are precisely the ordinals.

- (a) Show that $\mathcal{O}$ is an ordinal.
- (b) Hence derive a contradiction.

The Burali-Forti Paradox

Exercise 8.7

Suppose that there is a set $\mathcal{O}$ whose members are precisely the ordinals.

- (a) Show that $\mathcal{O}$ is an ordinal.
- (b) Hence derive a contradiction.

Solution

In the whiteboard.

The Burali-Forti Paradox

Historical remark

*“This contradiction is called **Burali-Forti’s paradox**, published in 1897 by the Italian Cesare Burali-Forti [...] It was the **first** key paradox to arise from Cantor’s work and sparked the proper evaluation of the foundations of set theory. Its significance in the context of ZF is that the ordinals form a **proper class**.” (p. 208)*

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