MT8002 Elements of Set Theory
7.3 Ordered Sets. Order Arithmetic

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

Outline

Introduction

We shall define two arithmetic operations on linearly ordered sets: the order sum and the order product.

Outline

Order Sum

Definition (p. 51)

Let $\mathbf{m}$ and $\mathbf{n}$ be natural numbers. The (arithmetic) **addition** of $\mathbf{m}$ and $\mathbf{n}$, denoted $\mathbf{m}+\mathbf{n}$, is defined by recursion by

$$\begin{aligned} \mathbf{m}+\mathbf{0} &:= \mathbf{m}, \\ \mathbf{m}+\mathbf{n}^{+} &:= (\mathbf{m}+\mathbf{n})^{+}. \end{aligned}$$

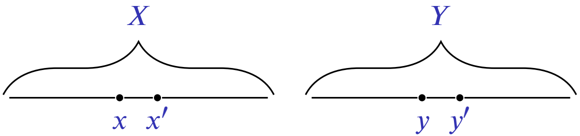
Order Sum

Let $\mathbf{m}$ and $\mathbf{n}$ be natural numbers. When $\mathbf{m}$ and $\mathbf{n}$ are regarded as strictly linearly ordered sets by the relation $\in$, one natural goal is to define their addition, in the order-theoretic sense, such that it is order-isomorphic to their arithmetic addition $\mathbf{m} + \mathbf{n}$.

Order Sum

Intuitive idea

Let $(X, <_X)$ and $(Y, <_Y)$ be two linearly ordered sets. An intuitive idea for adding the sets is the following one.



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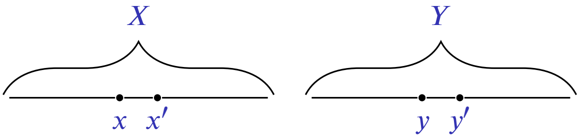

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A technical issue

What about if the sets X and Y have elements in common?

Order Sum

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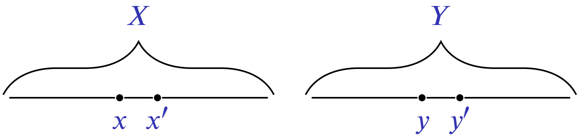


$$\cdots x < x' < \cdots < y < y' \cdots$$

A technical issue

What about if the sets X and Y have elements in common?
Solution: To use disjoint copies of X and Y.

Order Sum

Definition (p. 177)

Let $(X, <_X)$ and $(Y, <_Y)$ be two linearly ordered sets. The **order sum** of X and Y, denoted $X + Y$, is the strictly linearly ordered set

$$(X \times \{\mathbf{0}\}) \cup (Y \times \{\mathbf{1}\})$$

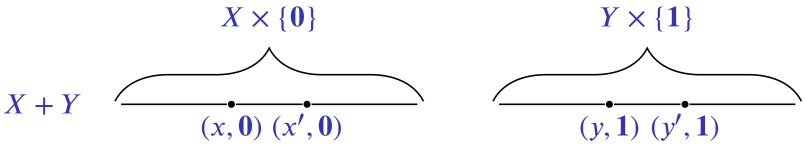
by the relation $<$ defined by

$$\begin{aligned}
&(x, \mathbf{0}) < (x', \mathbf{0}) \text{ if } x, x' \in X \text{ and } x <_X x'; \\
&(x, \mathbf{0}) < (y, \mathbf{1}) \text{ if } x \in X \text{ and } y \in Y; \\
&(y, \mathbf{1}) < (y', \mathbf{1}) \text{ if } y, y' \in Y \text{ and } y <_Y y'.
\end{aligned}$$

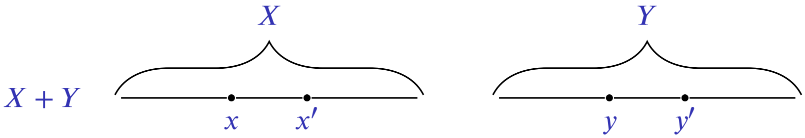
Order Sum

Representation

(a) From the definition



(b) From the textbook



Order Sum

Question

Let $(X, <_X)$ and $(Y, <_Y)$ be two linearly ordered sets. Is always $(X, <_X) + (Y, <_Y)$ a "new" set? That is,

$$(X, <_X) \ncong (X, <_X) + (Y, <_Y) \quad \text{and} \quad (Y, <_Y) \ncong (X, <_X) + (Y, <_Y)$$

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Answer

(a) Finite (non-empty) linearly ordered sets: Yes, always.

(b) Infinite linearly ordered sets: No, not always.

Order Sum

In the following theorem we shall denote the arithmetic addition of natural numbers by $\oplus$.

Order Sum

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Theorem 7.4

Let $\mathbf{m}$ and $\mathbf{n}$ be natural numbers. Then their order-theoretic addition $\mathbf{m} + \mathbf{n}$ is order-isomorphic to their arithmetic addition $\mathbf{m} \oplus \mathbf{n}$. That is, $(\mathbf{m}, \in) + (\mathbf{n}, \in) \cong (\mathbf{m} \oplus \mathbf{n}, <)$.

(continued on next slide)

Order Sum

Proof.

Let $p(n) : \mathbf{m} + n \cong \mathbf{m} \oplus n$ be a property where n is a free variable and $\mathbf{m}$ is a fixed natural number (parameter).

Order Sum

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The proof is by induction on $\mathbf{n}$, for fixed $\mathbf{m}$ (recall that arithmetic addition was defined by recursion in the second argument).

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The proof is by induction on $\mathbf{n}$, for fixed $\mathbf{m}$ (recall that arithmetic addition was defined by recursion in the second argument).

(1) Basis step $p(\mathbf{0})$: Since $\mathbf{m} + \mathbf{0} \cong \mathbf{m}$ and $\mathbf{m} \cong \mathbf{m} \oplus \mathbf{0}$, then $\mathbf{m} + \mathbf{0} \cong \mathbf{m} \oplus \mathbf{0}$.

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Let $p(n) : \mathbf{m} + n \cong \mathbf{m} \oplus n$ be a property where n is a free variable and $\mathbf{m}$ is a fixed natural number (parameter).

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(1) Basis step $p(\mathbf{0})$: Since $\mathbf{m} + \mathbf{0} \cong \mathbf{m}$ and $\mathbf{m} \cong \mathbf{m} \oplus \mathbf{0}$, then $\mathbf{m} + \mathbf{0} \cong \mathbf{m} \oplus \mathbf{0}$.

(2) Inductive step, for all $\mathbf{n}$, $p(\mathbf{n}) \to p(\mathbf{n}^+)$: The order-isomorphism $f : \mathbf{m} + \mathbf{n} \longrightarrow \mathbf{m} \oplus \mathbf{n}$ from the inductive hypothesis should be extended to an order-isomorphism from $\mathbf{m} + \mathbf{n}^+$ to $\mathbf{m} \oplus \mathbf{n}^+$, using $\mathbf{n}^+ \doteq \mathbf{n} \cup \{\mathbf{n}\}$ to send the extra element of each side to the other.

■

Order Sum

Definition (p. 179)

ω (omega) $:= (\mathbb{N}, \in)$.

Order Sum

Example

Are $\omega + \mathbf{2}$ and $\mathbf{2} + \omega$ order-isomorphic? (Exercise 7.27).

Order Sum

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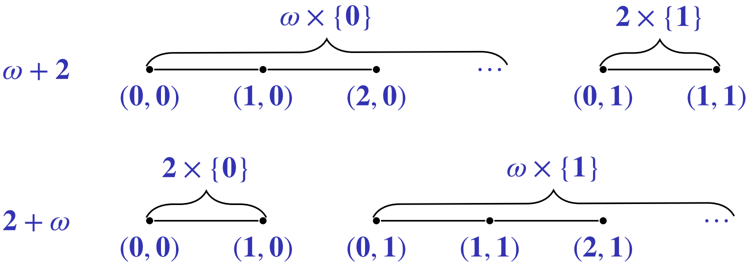
Are $\omega + \mathbf{2}$ and $\mathbf{2} + \omega$ order-isomorphic? (Exercise 7.27).

No! The element $(\mathbf{0}, \mathbf{1})$ is a limit point of $\omega + \mathbf{2}$, but $\mathbf{2} + \omega$ has no limit points, so $\omega + \mathbf{2} \not\cong \mathbf{2} + \omega$.



Order Sum

Remark

From the previous example we know that order sum is not commutative respect to order-isomorphism.

Order Sum

Example

Are ω and $\mathbf{2} + \omega$ order-isomorphic? (Exercise 7.27).

Order Sum

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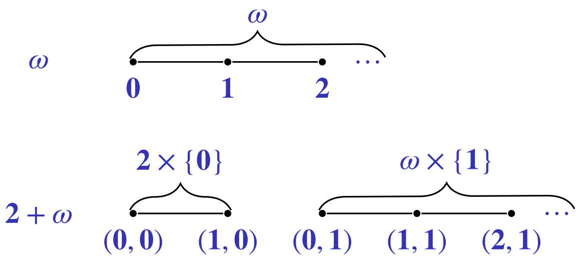
Yes! $\omega \cong \mathbf{2} + \omega$.

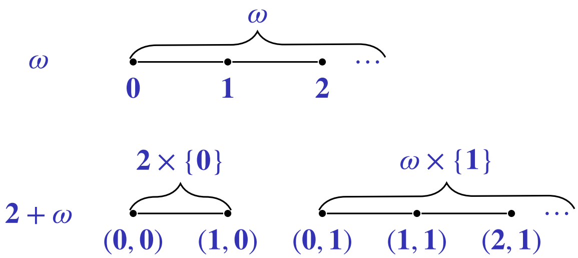


Order Sum

Example

Are ω and $\mathbf{2}+\omega$ order-isomorphic? (Exercise 7.27).

Yes! $\omega \cong \mathbf{2}+\omega$.



The order-isomorphism is the function defined by

$$f : \omega \longrightarrow \mathbf{2}+\omega$$

$$\mathbf{i} \longmapsto \begin{cases} (\mathbf{i}, \mathbf{0}), & \text{if } \mathbf{i} \doteq \mathbf{0} \text{ or } \mathbf{1}; \\ (\mathbf{i}-\mathbf{2}, \mathbf{1}), & \text{if } \mathbf{i} \geq \mathbf{2}. \end{cases}$$

Order Sum

Example

(a) $\omega \cong \mathbf{i} + \omega,$ for $\mathbf{i} \in \omega$.

(b) $\omega \ncong \omega + \mathbf{i},$ for $\mathbf{i} \in \omega$.

(c) $\omega + \mathbf{1} \ncong \omega + \mathbf{2} \ncong \omega + \mathbf{3} \ncong \cdots \ncong \omega + \mathbf{n} \cdots$.

Order Sum

Exercise 7.31

For each of the following ordered sets X define a new order $<'$ on the set $\mathbb{N}$ so that with this order the set $\mathbb{N}$ is order-isomorphic to the X.

(a) $X = \omega + \mathbf{2}$,

(b) $X = \omega + \omega$.

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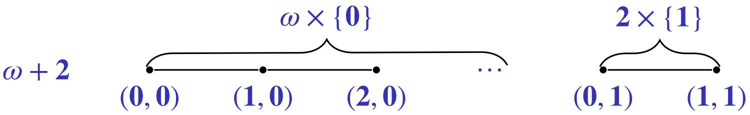
Order Sum

Solution

For $X = \omega + \mathbf{2}$.



On $\mathbb{N}$ we define $<'$ so that

$$2 <' 3 <' 4 <' \cdots <' n <' \cdots <' 0 <' 1.$$

(continued on next slide)

Order Sum

Solution

For $X = \omega + \omega$.

$$\omega+\omega \qquad \overbrace{(\mathbf{0},\mathbf{0}) \quad (\mathbf{1},\mathbf{0}) \quad (\mathbf{2},\mathbf{0}) \quad \cdots}^{\omega\times\{\mathbf{0}\}} \qquad \overbrace{(\mathbf{0},\mathbf{1}) \quad (\mathbf{1},\mathbf{1}) \quad (\mathbf{2},\mathbf{1}) \quad \cdots}^{\omega\times\{\mathbf{1}\}}$$

On $\mathbb{N}$ we define $<'$ so that

$$0 <' 2 <' 4 <' \cdots <' 2n <' \cdots <' 1 <' 3 <' 5 <' \cdots <' 2n+1 <' \cdots .$$

Outline

Order Product

Definition (p. 51)

Let $\mathbf{m}$ and $\mathbf{n}$ be natural numbers. The (arithmetic) **multiplication** of $\mathbf{m}$ and $\mathbf{n}$, denoted $\mathbf{m} \cdot \mathbf{n}$, is defined by recursion by

$$\begin{aligned} \mathbf{m} \cdot \mathbf{0} &:= \mathbf{0}, \\ \mathbf{m} \cdot \mathbf{n}^{+} &:= (\mathbf{m} \cdot \mathbf{n}) + \mathbf{m}. \end{aligned}$$

Order Product

Let $\mathbf{m}$ and $\mathbf{n}$ be natural numbers. When $\mathbf{m}$ and $\mathbf{n}$ are regarded as strictly linearly ordered sets by the relation $\in$, one natural goal is to define their multiplication, in the order-theoretic sense, such that it is order-isomorphic to their arithmetic multiplication $\mathbf{m} \cdot \mathbf{n}$.

Order Product

Definition (p. 182)

Let $(X, <_X)$ and $(Y, <_Y)$ be linearly ordered sets. The **lexicographic order** on $X \times Y$, denoted $<$, is defined by

$$(x, y) < (x', y') := \begin{cases} x <_X x', \text{ or} \\ x \doteq x' \text{ and } y <_Y y'. \end{cases}$$

Order Product

Definition (p. 182)

Let $(X, <_X)$ and $(Y, <_Y)$ be linearly ordered sets. The **anti-lexicographic order** on $X \times Y$, denoted $<$, is defined by

$$(x, y) < (x', y') :\!= \begin{cases} y <_Y y', & \text{or} \\ y \doteq y' & \text{and } x <_X x'. \end{cases}$$

Order Product

Theorem

Let $(X, <_X)$ and $(Y, <_Y)$ be linearly ordered sets. The **anti-lexicographic order** on $X \times Y$ is a strict linear order (Exercise 7.33).

Order Product

Definition (p. 182)

Let $(X, <_X)$ and $(Y, <_Y)$ be two linearly ordered sets. The **(order) product** of X and Y, is the strictly linearly ordered set

$$(X \times Y, <),$$

where $<$ is the anti-lexicographic order.

Order Product

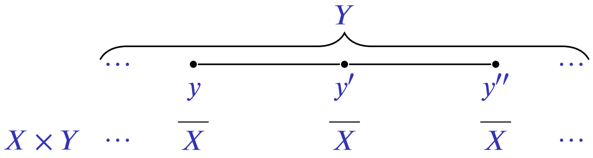
Convention

"*From now on, when we talk about the ordered set $X \times Y$ we shall assume that it has the anti-lexicographic order.*" *(p. 183)*

Order Product

Representation of $X \times Y$

The product $X \times Y$ "*is as though a copy of* X *is attached to each* $y \in Y$, *with the order increasing from left to right through these copies.*" *(p. 183)*



Order Product

Theorem

Let $\mathbf{m}$ and $\mathbf{n}$ be natural numbers. Then their order-theoretic product $\mathbf{m} \times \mathbf{n}$ is order-isomorphic to their arithmetic product $\mathbf{m} \cdot \mathbf{n}$, That is, $(\mathbf{m}, \in) \times (\mathbf{n}, \in) \cong (\mathbf{m} \cdot \mathbf{n}, \in)$ (Exercise 7.35(b)).

Order Product

Example

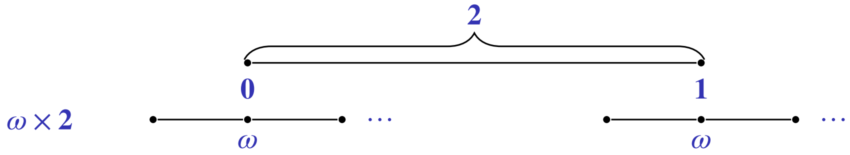
Are $\omega \times \mathbf{2}$ and $\omega + \omega$ order-isomorphic? (Exercise 7.36).

Order Product

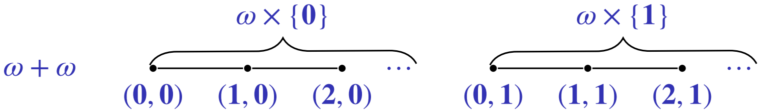
Example

Are $\omega \times \mathbf{2}$ and $\omega + \omega$ order-isomorphic? (Exercise 7.36).

Yes! The sets $\omega \times \mathbf{2}$ and $\omega + \omega$ are the same set with the same order.



$$(\mathbf{0},\mathbf{0}) < (\mathbf{1},\mathbf{0}) < (\mathbf{2},\mathbf{0}) < \cdots < (\mathbf{0},\mathbf{1}) < (\mathbf{1},\mathbf{1}) < (\mathbf{2},\mathbf{1}) < \cdots$$



Order Product

Remark

Let X be a strictly linearly order set. In general,

(a) $X \times \mathbf{2} \doteq X + X$ (so $X \times \mathbf{2} \cong X + X$),

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(b) $X \times \mathbf{3} \not\doteq X + X + X$, but

(c) $X \times \mathbf{3} \cong X + X + X$.

Order Product

Example

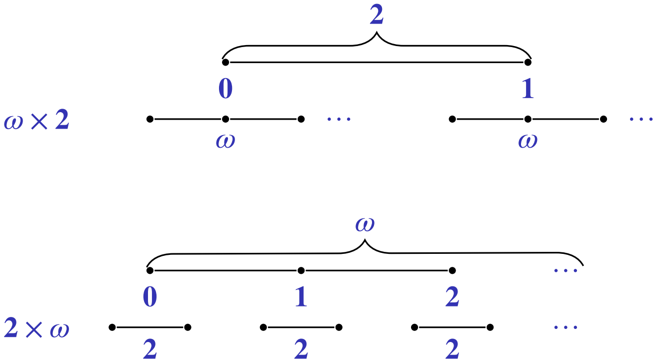
Are $\omega \times \mathbf{2}$ and $\mathbf{2} \times \omega$ order-isomorphic? (Exercise 7.36)

Order Product

Example

Are $\omega \times \mathbf{2}$ and $\mathbf{2} \times \omega$ order-isomorphic? (Exercise 7.36)

No! The element $(\mathbf{0}, \mathbf{1})$ is a limit point of $\omega \times \mathbf{2}$, but $\mathbf{2} \times w$ has not limit points.



Order Product

Example

Are $\mathbf{2} \times \omega$ and ω order-isomorphic? (Exercise 7.36).

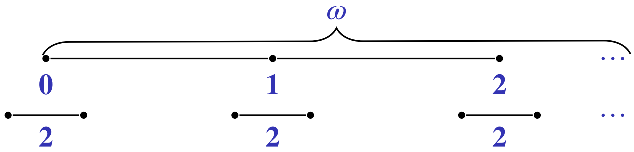
Order Product

Example

Are $\mathbf{2} \times \omega$ and ω order-isomorphic? (Exercise 7.36).

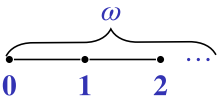
Yes! $\mathbf{2} \times \omega \cong \omega$.

$\mathbf{2} \times \omega$



$$(\mathbf{0}, \mathbf{0}) < (\mathbf{1}, \mathbf{0}) < (\mathbf{0}, \mathbf{1}) < (\mathbf{1}, \mathbf{1}) < (\mathbf{0}, \mathbf{2}) < (\mathbf{1}, \mathbf{2}) < \cdots$$

ω



Order Product

Remark

From the previous examples we know that order product is not commutative respect to order-isomorphism:

$$\omega + \omega \cong \omega \times 2 \ncong 2 \times \omega \cong w.$$

Order Sum

Example

(a) $\omega \cong \mathbf{i} \times \omega,$ for $\mathbf{i} \neq \mathbf{0}$ and $\mathbf{i} \in \omega$.

(b) $\omega \ncong \omega \times \mathbf{i},$ for $\mathbf{i} \neq \mathbf{1}$ and $\mathbf{i} \in \omega$.

(c) $\omega \times \mathbf{1} \ncong \omega \times \mathbf{2} \ncong \omega \times \mathbf{3} \ncong \cdots \ncong \omega \times \mathbf{n} \cdots$.

Order Product

Exercise 7.38

For each of the following ordered sets X define a new order $<'$ on the set $\mathbb{N}$ so that with this order the set $\mathbb{N}$ is order-isomorphic to the X.

(a) $X = \omega \times \mathbf{3}$,

(b) $X = (\omega \times \mathbf{3}) + \mathbf{4}$,

(c) $X = \omega \times \omega$,

(d) $X = (\omega \times \omega) + \omega$.

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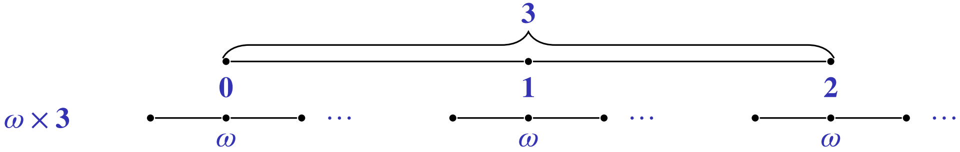
Order Product

Solution ($X = \omega \times \mathbf{3}$)



$$(\mathbf{0},\mathbf{0}) < (\mathbf{1},\mathbf{0}) < (\mathbf{2},\mathbf{0}) < \cdots < (\mathbf{0},\mathbf{1}) < (\mathbf{1},\mathbf{1}) < (\mathbf{2},\mathbf{1}) < \cdots < (\mathbf{0},\mathbf{2}) < (\mathbf{1},\mathbf{2}) < (\mathbf{2},\mathbf{2}) < \cdots$$

We partition $\mathbb{N}$ into three infinite subsets, each ordered by the usual $<$,

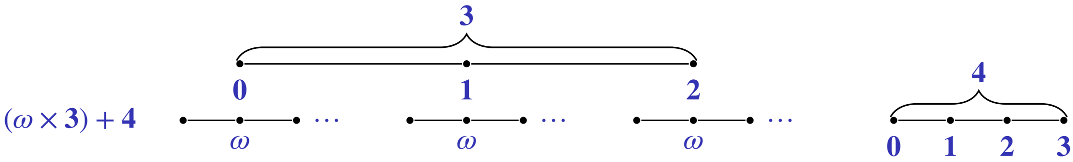
$$\{3n : n \in \mathbb{N}\}, \qquad \{3n+1 : n \in \mathbb{N}\}, \qquad \{3n+2 : n \in \mathbb{N}\},$$

and order these subsets by

$$0 <' 3 <' 6 <' \cdots <' 1 <' 4 <' 7 <' \cdots <' 2 <' 5 <' 8 <' \cdots$$

Order Product

Solution ($X = (\boldsymbol{\omega} \times \mathbf{3}) + \mathbf{4}$)



From the previous exercise we move the natural numbers $0, 1, 2, 3$ to the top of a new order $<'$:

$$6 <' 9 <' 12 <' \cdots <' 4 <' 7 <' 10 <' \cdots <' 5 <' 8 <' 11 <' \cdots <' 0 <' 1 <' 2 <' 3.$$

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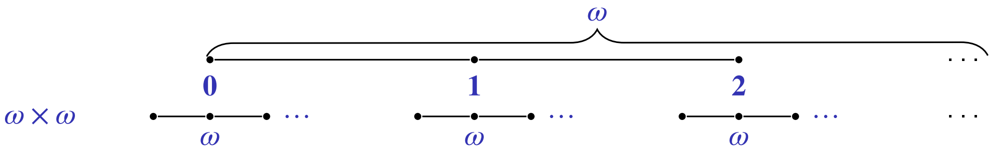
Order Product

Solution ($X = \omega \times \omega$)



$$(\mathbf{0},\mathbf{0}) < (\mathbf{1},\mathbf{0}) < (\mathbf{2},\mathbf{0}) < \cdots < (\mathbf{0},\mathbf{1}) < (\mathbf{1},\mathbf{1}) < (\mathbf{2},\mathbf{1}) < \cdots < (\mathbf{0},\mathbf{2}) < (\mathbf{1},\mathbf{2}) < (\mathbf{2},\mathbf{2}) < \cdots$$

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Order Product

Solution ($X = \omega \times \omega$, continuation)

We need to partition $\mathbb{N}$ into infinitely many infinite subsets $\{B_n : n \in \mathbb{N}\}$, order each B_n such that $B_n \cong \omega$, and also order the B_n themselves order-isomorphically to ω.

Solution ($X = \omega \times \omega$, continuation)

We need to partition $\mathbb{N}$ into infinitely many infinite subsets $\{B_n : n \in \mathbb{N}\}$, order each B_n such that $B_n \cong \omega$, and also order the B_n themselves order-isomorphically to ω.

Let p_n, for $n \geq 1$, be the n-th prime. We defined the subsets B_n by

$$B_n = \{p_n^i : i > 0\}, \text{ for } n \geq 1;$$
$$B_0 = \{0, 1\} \cup \{\text{missing composite numbers}\}.$$

Then the order $<'$ on $\mathbb{N}$ is defined by

$$i <' j := \begin{cases} i, j \in B_n \text{ for some } n \text{ and } i < j, \text{ or} \\ i \in B_m \text{ and } j \in B_n \text{ with } m < n. \end{cases}$$

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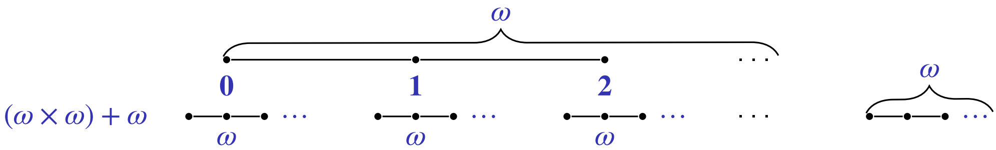
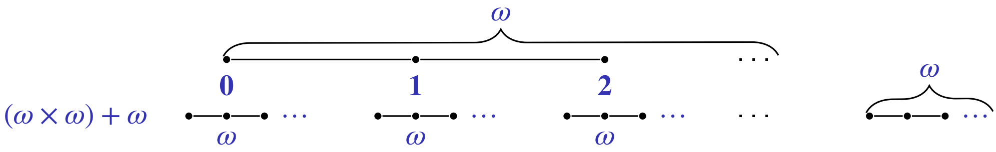
Order Product

Solution ($X = (\omega \times \omega) + \omega$)



Order Product

Solution ($X = (\omega \times \omega) + \omega$)



From the previous exercise we arranged B_0 such that every element of B_0 is greater than every element of every other B_n. That is, we define the order $<'$ on $\mathbb{N}$ by

$$i <' j := \begin{cases} i, j \in B_n \text{ for some } n \text{ and } i < j, \text{ or} \\ i \in B_m \text{ and } j \in B_n \text{ and } 0 < m < n, \text{ or} \\ i \in B_m \text{ for some } m > 0 \text{ and } j \in B_0. \end{cases}$$

Order Product

Question

Does order sum distribute over order product as in ordinary arithmetic?

Order Product

Question

Does order sum distribute over order product as in ordinary arithmetic?

Exercise 7.40

Let X, Y and Z be strictly linearly ordered sets.

(a) Show that $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$.

(b) Is it always the case that $(X + Y) \times Z \cong (X \times Z) + (Y \times Z)$?

Order Product

Notation

Let X be a linearly ordered set.

$X^n := \underbrace{X \times X \times \cdots \times X}_{n}$, for $n \in \mathbb{N}$ and $n > 0$.

Outline

References

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study. Chapman & Hall (cit. on p. 2).