MT8002 Elements of Set Theory
7.2 Ordered Sets. Linearly Ordered Sets

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

Outline

Introduction

- We can use binary relations to order some (partial order) or all (linear order) the elements of a set.
- Orders are binary relations that satisfy certain properties.
- Weak orders and strict orders can be interchangeably defined.

Outline

Binary Relations

Definition (p. 164)

Let X be a set. A set R is a **binary relation on** X if $R \subseteq X \times X$.

Notation

Let R be a binary relation on a set X.

(a) We write $(x, y) \in R$ or xRy, if $x \in X$ is related to $y \in X$ by the relation R.

(b) We write $(x, y) \notin R$ or $x \not{R} y$, if $x \in X$ is not related to $y \in X$ by the relation R.

Outline

Weak Orders

Definitions (p. 164)

A binary relation R on a set X is a **weak partial order** on the set X (or the set X is **weakly partially ordered** by the relation R), if it has the following properties:

(a) **reflexive**: for all $x \in X$, xRx;

(b) **anti-symmetric**: for all $x, y \in X$, if xRy and yRx then $x \doteq y$;

(c) **transitive**: for all $x, y, z \in X$, if xRy and yRz then xRz.

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(c) **transitive**: for all $x, y, z \in X$, if xRy and yRz then xRz.

If additionally the binary relation R has the following property:

(d) **linear**: for all $x, y \in X$, xRy or yRx,

then the relation R is a **weak linear order** on the set X (or the set X is **weakly linearly ordered** by the relation R).

Weak Orders

Examples (informal, p. 164)

The usual relations $\leq$ and $\geq$ on the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are weak linear orders on those sets.

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Example

Let X be a set. The relation $\subseteq$ on $\mathcal{P}(X)$ is a weak partial order.

Weak Orders

Exercise

Let X be a set. Is the relation $\subseteq$ on $\mathcal{P}(X)$ a weak linear order?

Weak Orders

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Let X be a set. Is the relation $\subseteq$ on $\mathcal{P}(X)$ a weak linear order?

Answer

No! For example the relation $\subseteq$ on the power set of $\{a, b\}$ is no linear because $\{a\}, \{b\} \in \mathcal{P}(\{a, b\})$ but neither $\{a\} \not\subseteq \{b\}$ not $\{b\} \not\subseteq \{a\}$.

Weak Orders

Description

The **restriction** of a weak partial order to a subset is the weak partial order obtained by discarding all comparisons involving elements outside of the subset.

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Definition (p. 164)

Let R be a weak partial order on a set X and let $A \subseteq X$. The **restriction** of the relation R to the set A, denoted $R|_{A\times A}$, is the relation defined by

$$R|_{A\times A} := \{(a,b) \in R : a, b \in A\}.$$

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Theorem (Exercise 7.2)

Let R be a weak partial (respectively linear) order on a set X and let $A \subseteq X$. Show that $R|_{A\times A}$ is a partial (respectively linear) order on the set A.

Outline

Strict Orders

Definitions (p. 165)

A binary relation S on a set X is a **strict partial order** on X (or the set X is **strictly partially ordered** by the relation S) if it has the following properties:

(a) **irreflexive**: for all $x \in X$, $x \not S x$;

(b) **transitive**: for all $x, y, z \in X$, if xSy and ySz then xSz.

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A binary relation S on a set X is a **strict partial order** on X (or the set X is **strictly partially ordered** by the relation S) if it has the following properties:

(a) **irreflexive**: for all $x \in X$, $x \not S x$;

(b) **transitive**: for all $x, y, z \in X$, if xSy and ySz then xSz.

If additionally the binary relation S has the following property:

(c) **linear**: for all $x, y \in X$, xSy or $x \doteq y$ or ySx,

then the relation S is a **strict linear order** on the set X (or the set X is **strictly linearly ordered** by the relation S).

Strict Orders

Examples

The usual relations $<$ and $>$ on the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are strict linear orders on those sets.

Example

Let X be a set. The relation $\subset$ on $\mathcal{P}(X)$ is a strict partial order.

Strict Orders

Theorem (relationship between weak and strict partial orders)

(a) Let R be a weak partial order on a set X. The binary relation S defined by

$$xSy := xRy \text{ and } x \neq y$$

is a strict partial order on the set X.

Strict Orders

Theorem (relationship between weak and strict partial orders)

(a) Let R be a weak partial order on a set X. The binary relation S defined by

$$xSy := xRy \text{ and } x \not\doteq y$$

is a strict partial order on the set X.

(b) Let S be a strict partial order on a set X. The binary relation R defined by

$$xRy := xSy \text{ or } x \doteq y$$

is a weak partial order on the set X.

Strict Orders

Theorem (relationship between weak and strict partial orders)

(a) Let R be a weak partial order on a set X. The binary relation S defined by

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is a strict partial order on the set X.

(b) Let S be a strict partial order on a set X. The binary relation R defined by

$$xRy := xSy \text{ or } x \doteq y$$

is a weak partial order on the set X.

See Exercise 7.4.

Outline

Orders on Natural Numbers

An “extra” property of natural numbers

$$\mathbf{0} \in \mathbf{1} \in \mathbf{2} \in \mathbf{3} \in \cdots$$

$$\mathbf{0} \in \{\mathbf{0}\} \in \{\mathbf{0}, \mathbf{1}\} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\} \in \cdots$$

$$\emptyset \in \{\emptyset\} \in \{\emptyset, \{\emptyset\}\} \in \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \in \cdots$$

Orders on Natural Numbers

Theorem 3.6

The set of natural numbers $\mathbb{N}$ is strictly linearly ordered by the binary relation $\in$.

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The set of natural numbers $\mathbb{N}$ is strictly linearly ordered by the binary relation $\in$.

That is, the binary relation $\in$ on $\mathbb{N}$ has the following properties:

(a) **irreflexive**: for all $\mathbf{n} \in \mathbb{N}$, $\mathbf{n} \notin \mathbf{n}$;

(b) **transitive**: for all $\mathbf{m}, \mathbf{n}, \mathbf{p} \in \mathbb{N}$, if $\mathbf{m} \in \mathbf{n}$ and $\mathbf{n} \in \mathbf{p}$ then $\mathbf{m} \in \mathbf{p}$;

(c) **linear**: for all $\mathbf{m}, \mathbf{n} \in \mathbb{N}$, $\mathbf{m} \in \mathbf{n}$ or $\mathbf{m} \doteq \mathbf{n}$ or $\mathbf{n} \in \mathbf{m}$.

Orders on Natural Numbers

Example (p. 42)

Although the binary relation $\in$ is transitive on the set $\mathbb{N}$ it does not mean that the relation is transitive on any set. For example, the relation is not transitive on the set

$$\{\mathbf{0}, \{\mathbf{0}\}, \{\{\mathbf{0}\}\}\}$$

because $\mathbf{0} \in \{\mathbf{0}\}$ and $\{\mathbf{0}\} \in \{\{\mathbf{0}\}\}$ but $\mathbf{0} \notin \{\{\mathbf{0}\}\}$.

Orders on Natural Numbers

Definition (p. 41)

For all $\mathbf{m}, \mathbf{n} \in \mathbb{N}$,

$\mathbf{m} < \mathbf{n} := \mathbf{m} \in \mathbf{n}$,
$\mathbf{m} \leq \mathbf{n} := \mathbf{m} < \mathbf{n}$ or $\mathbf{m} \doteq \mathbf{n}$.

Outline

Conventions

- "*We shall often write the order relation on X as $<_X$ or $\leq_X$, depending on whether the order is strict or weak.*" *(p. 166)*
- Sometimes a (weakly or strictly; partially or linearly) ordered set X by a relation R will we denote by the pair (X, R).[†]
- Joining above conventions we shall write:
  - "Let $(X, \leq_X)$ be a partially (respectively linearly) ordered set" when $\leq_X$ is a weak partial (respectively linear) order on the set X.
  - "Let $(X, <_X)$ be a partially (respectively linearly) ordered set" when $<_X$ is a strict partial (respectively linear) order on the set X.

[†]Note that the textbook does not use this convention.

Outline

Order-Isomorphism

Motivation

We need a precise way to express when two ordered sets are “the same”.

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Definitions (p. 166)

Let $(X, <_X)$ and $(Y, <_Y)$ be partially ordered sets. The ordered sets are **order-isomorphic**, denoted $X \cong Y$ or $(X, <_X) \cong (Y, <_Y)$, if there is a bijection $f : X \longrightarrow Y$ such that for all $x_1, x_2 \in X$,

$$x_1 <_X x_2 \quad \text{if and only if} \quad f(x_1) <_Y f(x_2).$$

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The map f is an **order-isomorphism**.

Remark

The above definition on weak partial orders is similar.

Order-Isomorphism

Exercise (informal)

Let $<$ and $>$ be the usual orders on $\mathbb{Z}$. Show that $(\mathbb{Z}, <)$ and $(\mathbb{Z}, >)$ are order-isomorphic.

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Solution

Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be the function $n \longmapsto -n$. We can show that the function f is a bijection and that it preserves the order, that is, for all $m, n \in \mathbb{Z}$,

$$m < n \quad \text{if and only if} \quad -m > -n.$$

Order-Isomorphism

Theorem

The binary relation $\cong$ is an equivalence relation on strictly partially ordered sets.

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That is, let $(X, <_X)$, $(Y, <_Y)$ and $(Z, <_Z)$ be partially ordered sets, then the binary relation $\cong$ satisfies the following properties (see Exercise 7.6):

(a) **reflexive**: for all X, $X \cong X$;

(b) **symmetric**: for all X and Y, if $X \cong Y$ then $Y \cong X$;

(c) **transitive**: for all X, Y and Z, if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

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Theorem

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(b) **symmetric**: for all X and Y, if $X \cong Y$ then $Y \cong X$;

(c) **transitive**: for all X, Y and Z, if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$.

Technical remark

Note that the binary relation $\cong$ should be defined on the universe $\mathscr{U}$.

Order-Isomorphism

Definition (p. 168)

An **order-theoretic invariant** is a property of an ordered set preserved under order-isomorphism.

Outline

Order-Embedding

Motivation

We also need to define when an ordered set fits inside another while preserving its order structure entirely.

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Definition (p. 167)

Let $(X, <_X)$ and $(Y, <_Y)$ be partially ordered sets. A function $f : X \longrightarrow Y$ is an **order-embedding** of X into Y (or of $(X, <_X)$ into $(Y, <_Y)$) if f is one-one and, for all $x_1, x_2 \in X$,

$$x_1 <_X x_2 \quad \text{if and only if} \quad f(x_1) <_Y f(x_2).$$

Order-Embedding

Exercise (informal)

Let $\mathbb{N}$ and $\mathbb{R}$ with the usual order $<$. Show an order-embedding of $(\mathbb{N}, <)$ into $(\mathbb{R}, <)$

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Solution

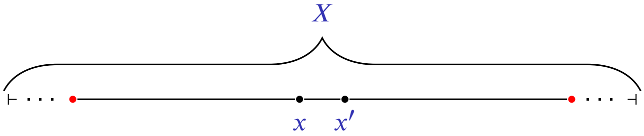
Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ be the function $n \longmapsto n$. We can show that the function f is one-one and that it preserves the order, that is, for all $m, n \in \mathbb{N}$,

$$m < n \quad \text{if and only if} \quad f(m) < f(n).$$

Outline

Representation of Linearly Ordered Sets

Let $(X, <_X)$ be a linearly ordered set and let $x, x' \in X$ with $x <_x x'$. We picture x to the left of x'. The red dots mean that an element does not belong to the set X.



"*Of course, we must not read too much into the picture. The picture is very suggestive of the usual real line, but for instance there might not be any elements of X between x and x' in the picture above.*" *(p. 167)*

Outline

Maximum and Minimum Elements

Definitions (p. 168)

Let $(X, \leq_X)$ be a linearly ordered set and let $a, b \in X$.

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Definitions (p. 168)

Let $(X, \leq_X)$ be a linearly ordered set and let $a, b \in X$.

(a) The element b is the **maximum** (or **greatest**) element of X, if $x \leq_X b$ for all $x \in X$.

Maximum and Minimum Elements

Definitions (p. 168)

Let $(X, \leq_X)$ be a linearly ordered set and let $a, b \in X$.

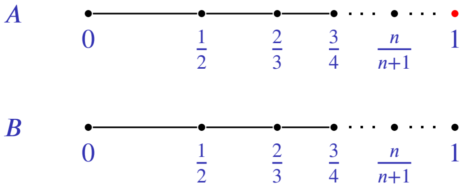
(a) The element b is the **maximum** (or **greatest**) element of X, if $x \leq_X b$ for all $x \in X$.

(b) The element a is the **minimum** (or **least**) element of X, if $a \leq_X x$ for all $x \in X$.

Maximum and Minimum Elements

Example (informal, p. 168)

Let $A = \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$ and $B = A \cup \{1\}$, subsets of $\mathbb{Q}$, with the usual order $\leq$.



The set A has minimum element, 0, but no maximum element.
The set B has minimum element, 0, and maximum element, 1.

Maximum and Minimum Elements

Remark

In the next theorem we shall see that the maximum and the minimum are order-theoretic invariants.

Maximum and Minimum Elements

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Theorem 7.1

Let $(X, \leq_X)$ and $(Y, \leq_X)$ be linearly ordered sets and let $f : X \longrightarrow Y$ be an order-isomorphism from X to Y. Then

(a) if X has a maximum element then so does Y;

(b) if X has a minimum element then so does Y.

Maximum and Minimum Elements

Example (informal)

Let $A = \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$ and $B = A \cup \{1\}$, subsets of $\mathbb{Q}$, with the usual order $\leq$.

By Theorem 7.1 the ordered sets $(A, \leq)$ and $(B, \leq)$ are not order-isomorphic because the set B has maximum element, but the set A does not.

Outline

Limit Points

Definition (p. 169)

Let $(X, <_X)$ be a linearly ordered set and let $c \in X$.
The element c is a **limit point** of X if

(a) there exits $x \in X$ such that $x <_X c$ and

(b) for all $x \in X$ with $x <_X c$, there exits $y \in X$ such that $x <_X y <_X c$.

Limit Points

Remark

In the next theorem we shall see that limit points are order-theoretic invariants.

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Theorem 7.2

Let $(X, <_X)$ and $(Y, <_X)$ be linearly ordered sets, let $f : X \longrightarrow Y$ be an order-isomorphism from X to Y and let $c \in X$. If c is a limit point of X then $f(c)$ is a limit point of Y (see Exercise 7.11).

Exercise (informal, p. 170)

Let A and D be subsets of $\mathbb{Q}$, with the usual order $<$, defined by

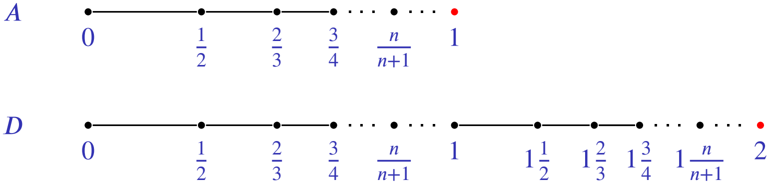
$$A = \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\},$$
$$D = \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\} \cup \left\{2 - \frac{1}{n+1} : n \in \mathbb{N}\right\}.$$

Are $(A, <)$ and $(D, <)$ order-isomorphic?

(continued on next slide)

Limit Points

Solution



(a) The element $0 = 1 - \frac{1}{0+1}$ is the minimum (so 0 is no a limit point) of the sets A and D.

(b) The set A has not limit points.

(c) The element $1 = 2 - \frac{1}{0+1}$ is a limit point of set D.

Therefore, since $(B, <)$ has one limit point but $(A, <)$ does not have, the ordered sets are not order-isomorphic by Theorem 7.2.

Outline

Successors and Predecessors

Definition (p. 170)

Let $(X, \leq_X)$ be a linearly ordered set and let $x, c \in X$. Then c is the **successor** of x, denoted x^+, and x is the **predecessor** of c, if

(a) $x <_X c$ and

(b) for all $y \in X$, if $x <_X y$ then $c \doteq y$ or $c <_X y$.

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(a) $x <_X c$ and

(b) for all $y \in X$, if $x <_X y$ then $c \doteq y$ or $c <_X y$.

That is, c is the successor of x (and x is the predecessor of c) if c is the first element of X greater than x.

Successors and Predecessors

Questions

In a strictly linearly ordered set

(a) can a limit point also be a successor?

Successors and Predecessors

Questions

In a strictly linearly ordered set

(a) can a limit point also be a successor?

(b) can a successor also be a limit point?

Successors and Predecessors

Examples (informal)

(a) In $\mathbb{R}$ with the usual order $<$, every real number is a limit point, so $\mathbb{R}$ has no successors.

Successors and Predecessors

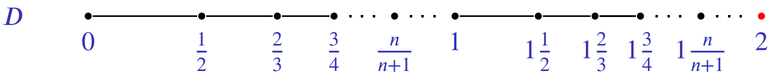
Examples (informal)

(a) In $\mathbb{R}$ with the usual order $<$, every real number is a limit point, so $\mathbb{R}$ has no successors.

(b) In $\mathbb{N}$ with the usual order $<$, every natural number different from zero is a successor, so $\mathbb{N}$ has no limit points.

Example

Let $D = \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\} \cup \left\{2 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$ be a subset of $\mathbb{Q}$, with the usual order $\leq$.



(a) The element $0 = 1 - \frac{1}{0+1}$ is the minimum (0 is neither a successor nor a limit point).

(b) Every element $1 - \frac{1}{n+1}$, with $n > 0$, is the successor of the element $1 - \frac{1}{n}$.

(c) The element $1 = 2 - \frac{1}{0+1}$ is a limit point.

(d) Note that for all $x \in D$, with $x < 1$, x has a successor x^+ and the set $\{x \in D : x < 1\}$ is infinite. See Exercise 7.14.

(e) Every element $2 - \frac{1}{n+1}$, with $n > 0$, is the successor of the element $2 - \frac{1}{n}$.

Outline

References

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study. Chapman & Hall (cit. on p. 2).