MT8002 Elements of Set Theory
The Zermelo–Fraenkel Axioms
Introduction

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Conventions

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

See the appendix in the course homepage for the logical symbols, mathematical symbols and acronyms not defined in these slides.

Outline

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Defining Sets by Extension and by Intension

Description

In naive set theory we can define sets by **extension** (explicitly listing its elements) or by **intension** (given a property of its elements).

Defining Sets by Extension and by Intension

Example (finite set)

$$\{2, 4, 6, 8, 10, 12, 14\}, \qquad \text{(by extension)};$$

$$\{x \in \mathbb{N} : 0 < x < 15 \text{ and } x \text{ is even}\}, \qquad \text{(by intension)}.$$

Defining Sets by Extension and by Intension

Example (finite set)

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Example (infinite set)

The infinite sets are defined by intension via a finite description.

$$\{\ldots, -4, -2, 0, 2, 4, \ldots\}, \qquad \text{(by extension)};$$

$$\{x \in \mathbb{Z} : x \text{ is even}\}, \qquad \text{(by intension)}.$$

Defining Sets by Extension and by Intension

Example

What about sets without an “elegant” finite description? No problem!

$$\{1, 7, 42\}, \qquad \text{(by extension);}$$
$$\{x : x = 1 \text{ or } x = 7 \text{ or } x = 42\}, \qquad \text{(by intension).}$$

Defining Sets by Extension and by Intension

A key feature of the set theory and mathematics [...] is that every mathematical object that we discuss can indeed be described in a finite way. As the most exciting part of set theory deals with infinite sets, and as such a set can never be finitely described by extension, definition by intension is crucial. (p. 66)

Defining Sets by Extension and by Intension

Description

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In the literature the above description is called the **Unrestricted (or Naive) Comprehension Principle**.

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† See, i.e.,, (Enderton 1977; Button 2021).

Defining Sets by Extension and by Intension

Example

Defining a set by intension allows some circularities.

Let $[\![1]\!]$ be the set defined by

$$[\![1]\!] = \{\, x : x \text{ has one element} \,\},$$

and be the singleton set $\{\, [\![1]\!] \,\}$. Then

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Question

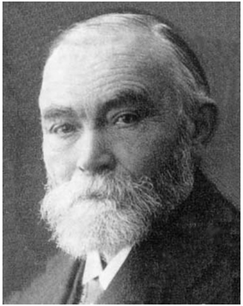
Is there any problem with the above circularity?

Outline

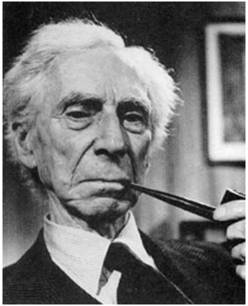
Russell's Paradox

Historical remark

The paradox was communicated in 1901 by the British mathematician and philosopher Bertrand Russell to the German mathematician and philosopher Gottlob Frege, undermining Frege's attempt to reduce mathematics to logic (images from Wikipedia)



Gottlob Frege (1848 – 1925)



Bertrand Russell (1872 – 1970)

Russell's Paradox

Description

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Does $\mathscr{R} \in \mathscr{R}$?

- If $\mathscr{R} \in \mathscr{R}$ then $\mathscr{R} \notin \mathscr{R}$.
- If $\mathscr{R} \notin \mathscr{R}$ then $\mathscr{R} \in \mathscr{R}$.

Russell's Paradox

Consequences

Recall from the previous lecture than Russell's paradox (and other paradoxes) generated a crisis in the foundations of the mathematics.

Russell's Paradox

Possible solutions

- Hierarchy of types (Russell and Whitehead)
- Restrict the ways in which construct new sets via the ZFC theory (Zermelo, Fraenkel, Skolem and other).

Carta of Russell a van Heijenoort

Penrhyndeudraeth, 23 de noviembre de 1962
Estimado Profesor van Heijenoort,

Cuando pienso en actos de gracia e integridad, me doy cuenta de que no conozco ninguno comparable con la dedicación de Frege a la verdad. Estaba Frege dando cima a la obra de toda su vida, la mayor parte de su trabajo había sido ignorado en beneficio de hombres infinitamente menos competentes que él, su segundo volumen estaba a punto de ser publicado y, al darse cuenta de que su supuesto fundamental era erróneo, reaccionó con placer intelectual, reprimiendo todo sentimiento de decepción personal. Era algo casi sobrehumano y un índice de aquello de lo que los hombres son capaces cuando están dedicados al trabajo creador y al conocimiento, y no al crudo afán por dominar y hacerse famosos.

Cordialmente,
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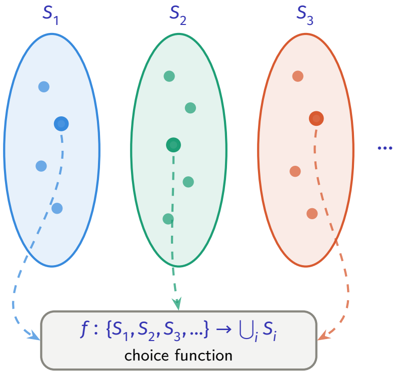
[†] La carta fue escrita en 1962. La versión original en inglés está en (van Heijenoort 1967, pág. 127). La versión en español está en (Frege [1971] 1973, págs. 8–9). La carta fue traducida por Jesús Mosterín.

Outline

The Axiom of Choice

Description

From any collection (finite or infinite) of non-empty sets, there exists a choice function that picks one element from each set.



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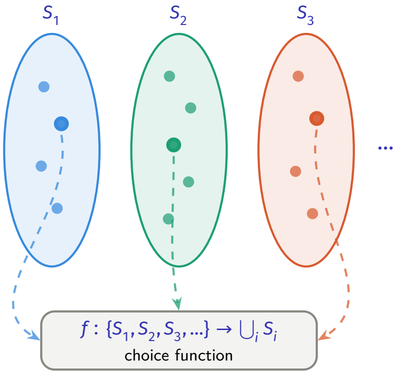


† LaTeX figure generated by Claude Code.

The Axiom of Choice

Remark

A very complete history of origins, development and influence of the axiom of choice is in (Moore 1982).

About the ZFC theory and the axiom of choice

> *"The axioms are based on those proposed by the German mathematician Ernst Zermelo (1871-1953) in 1905. It is important to realize, however, that Zermelo's motivation for producing axioms was not so much the avoidance of paradoxes, but primarily to give a framework for a controversial result about sets, of much interest to mathematicians of the day, and still of great significance. This result concerned a problem of Cantor's, whether it was possible to define a well-ordering on the set of real numbers, and Zermelo's argument, exploiting a principle called the axiom of choice, had previously been rejected by other mathematicians. By enunciating the assumptions about sets underlying his argument, Zermelo hoped to make it harder for others to reject it." (pp. 69–70)*

Outline

About the Theory of Zermelo–Fraenkel with Choice

Historical remark

The current ZFC is grounded mainly in the following works:

(Zermelo [1908] 1967),
(Fraenkel [1922] 1967),
(Skolem [1922] 1967),
(von Neumann [1923] 1967) and
(Zermelo [1930] 2010).

Ernst Zermelo
(1871–1853)

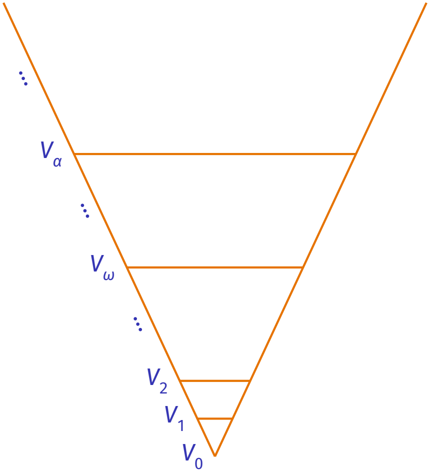
Adolf Fraenkel
(1891–1965)

Thoralf Skolem
(1887–1863)

John von Neumann
(1903–1957)

(Images from Wikipedia)

About the Theory of Zermelo–Fraenkel with Choice



$$\begin{aligned} V_0 &:= \varnothing, \\ V_{\alpha+1} &:= \mathscr{P}(V_\alpha), \text{ for any ordinal } \alpha, \\ V_\beta &:= \bigcup_{\alpha<\beta} V_\alpha, \text{ for any limit ordinal } \beta. \end{aligned}$$

Cumulative Hierarchy of Sets

About the Theory of Zermelo–Fraenkel with Choice

Definition (informal, p. 69)

There is not "set of all sets" as illustrated from the cumulative hierarchy of sets. However, sometimes we need to talk about this kind of collections.

A **class** is any object of the form $\{ x : x \text{ is a set with property } P \}$.

$$\text{Class} \begin{cases} \text{Proper class} \\ \text{Set} \end{cases}$$

A **proper class** is a class that "*cannot be treated as a set without entailing a contradiction*". *(p. 70)*

Informally, a proper class is a class "too large" to be a set.

About the Theory of Zermelo–Fraenkel with Choice

Example

The **universe** of sets, denoted $\mathscr{V}$, defined by

$$\mathscr{V} := \{\, x : x \text{ is a set} \,\}$$

is a proper class.

Outline

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