MT8002 Elements of Set Theory
6. Cardinals without the Axiom of Choice

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Semester 2026-2

Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

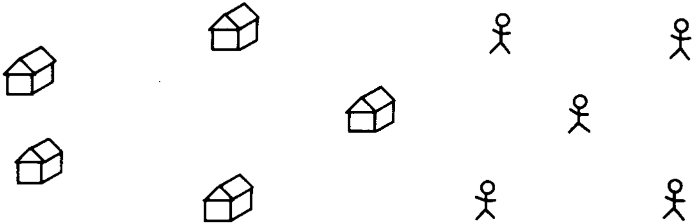
Outline

Outline

Introduction

Question

Suppose you only know how to count to three. Are the set of houses and the set of people in the figure the same size?†

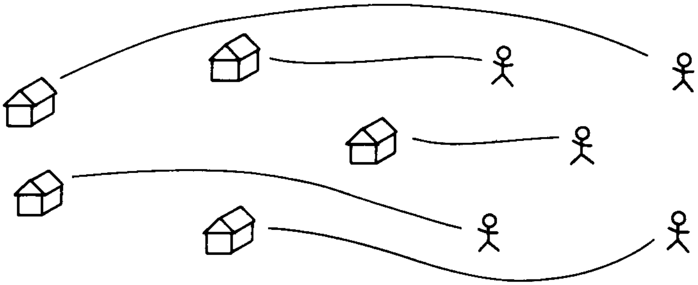


†Figure source: (Enderton 1977, Fig. 30).

Introduction

Answer

Yes! The set of houses is the same size as the set of people, as shown in the figure.[†]



[†]Figure source (Enderton 1977, Fig. 31).

Introduction

Question

In the previous question, the sets were finite. Can we apply similar reasoning if the sets are infinite?

Introduction

Definitions (p. 127)

(a) A set X is **finite** if there is a bijective function $f : \mathbf{n} \longrightarrow X$, for some natural number $\mathbf{n}$.

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(a) A set X is **finite** if there is a bijective function $f : \mathbf{n} \longrightarrow X$, for some natural number $\mathbf{n}$.

(b) A set X is **infinite** if it is not finite. That is, there is no a bijective function $f : \mathbf{n} \longrightarrow X$, for any natural number $\mathbf{n}$.

Outline

Comparing Sizes

Definitions (p. 130)

(a) A set A is **equinumerous** with a set B, denoted $A \approx B$, if there is a bijection $f : A \longrightarrow B$.

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Comparing Sizes

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(b) A set A is **less than or equinumerous** with a set B, denoted $A \precsim B$, if there is a one-to-one function $f : A \longrightarrow B$.

(c) A set A is **dominated** by a set B, denoted $A \prec B$, if$A \precsim B$ and it is not the case that $A \approx B$.

Comparing Sizes

Notation

$X \not\approx Y := \neg(X \approx Y)$,
$X \not\precsim Y := \neg(X \precsim Y)$,
$X \nprec Y := \neg(X \prec Y)$.

Comparing Sizes

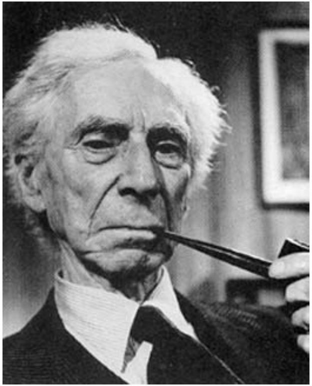
Examples

(a) $\mathbb{N} \precsim \mathbb{Z}$ and $\mathbb{Z} \precsim \mathbb{N}$.

(b) $\mathbb{N} \approx \mathbb{Z}$.

(c) $\mathbb{N} \not\prec \mathbb{Z}$ and $\mathbb{Z} \not\prec \mathbb{N}$.

Comparing Sizes



(1872 – 1970)

"*The possibility that whole and part may have the same number of terms is, it must be confessed, shocking to common-sense.*" *(Russell 1903, p. 358)*

Outline

Infinite Sets without AC - Countable Sets

Definitions (p. 143)

(a) A set A is **countable** if it is finite or equinumerous with $\mathbb{N}$.

Infinite Sets without AC - Countable Sets

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(a) A set A is **countable** if it is finite or equinumerous with $\mathbb{N}$.

(b) A set A is **countably infinite** if it is equinumerous with $\mathbb{N}$.

Infinite Sets without AC - Countable Sets

Examples

(a) Let $\mathbf{n} \in \mathbb{N}$. The set $\mathbf{n}$ is countable because it has n elements.

(b) The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ are countably infinite.

Infinite Sets without AC - Countable Sets

Theorem 6.5 (Cantor's Theorem)

For any set X, $X \prec \mathcal{P}(X)$.

Example

$\mathbb{N} \prec \mathcal{P}(\mathbb{N}) \prec \mathcal{P}(\mathcal{P}(\mathbb{N})) \prec \cdots$.

Infinite Sets without AC - Countable Sets

Definition (p. 148)

A set **uncountable** or **uncountably infinite** is an infinite set which is not countable.

Outline

Uncountable Sets and Cardinal Arithmetic, without AC

Remark

Recall that $\mathbf{2}^X \doteq \{\mathbf{0}, \mathbf{1}\}^X$ is the set of functions from the set X to $\{\mathbf{0}, \mathbf{1}\}$.

Uncountable Sets and Cardinal Arithmetic, without AC

Remark

Recall that $\mathbf{2}^X \doteq \{\mathbf{0},\mathbf{1}\}^X$ is the set of functions from the set X to $\{\mathbf{0},\mathbf{1}\}$.

Theorem 6.7

For any set X, $\mathcal{P}(X) \approx \mathbf{2}^X$.

Theorem 6.8

The set $\mathbb{R}$ is uncountable.

Theorem

$\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx \mathbf{2}^{\mathbb{N}}$.

Outline

References

Herbert B. Enderton (1977). Elements of Set Theory. Academic Press (cit. on pp. 5, 6).

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study. Chapman & Hall (cit. on p. 2).

Bertrand Russell (1903). The Principles of Mathematics. Cambridge University Press (cit. on p. 16).