MT8002 Elements of Set Theory
The Zermelo–Fraenkel Axioms
Axioms 7 to 9

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Semester 2026-2

Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Conventions

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

See the appendix in the course homepage for the logical symbols, mathematical symbols and acronyms not defined in these slides.

Outline

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Introduction

Formal language

Recall that in ZFC the concepts of pure set and membership relation are primitive (undefined). Also recall that the universe of discourse of the theory is the pure sets.

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Classification

- Axioms stating the existence of sets
- Axioms determining properties of sets
- Axioms for building sets from other sets

Outline

Inductive Sets

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Example

$$\begin{aligned} \varnothing^+ &\doteq \varnothing \cup \{ \varnothing \} \\ &\doteq \{ \varnothing \}, \end{aligned} \qquad \begin{aligned} \varnothing^{++} &\doteq (\varnothing^+)^+ \\ &\doteq \{ \varnothing \}^+ \\ &\doteq \{ \varnothing \} \cup \{ \{ \varnothing \} \} \\ &\doteq \{ \varnothing, \{ \varnothing \} \}, \end{aligned} \qquad \begin{aligned} \varnothing^{+++} &\doteq (\varnothing^{++})^+ \\ &\doteq \{ \varnothing, \{ \varnothing \} \}^+ \\ &\doteq \{ \varnothing, \{ \varnothing \} \} \cup \{ \{ \varnothing, \{ \varnothing \} \} \} \\ &\doteq \{ \varnothing, \{ \varnothing \}, \{ \varnothing, \{ \varnothing \} \} \}. \end{aligned}$$

Inductive Sets

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Remark

Note that an inductive set is an infinite set.

Outline

The Axiom of Infinite

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$\exists x\,(\emptyset \in x \wedge \forall y\,(y \in x \rightarrow y \cup \{y\} \in x))$.

The Axiom of Infinite

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$\exists x\,(\varnothing \in x \wedge \forall y\,(y \in x \rightarrow y \cup \{y\} \in x))$.

Informal description

"*There is an inductive set.*" *(p. 92)*

Category

Axiom stating the existence of set.

The Axiom of Infinite

Example

Let x be an inductive set by the Axiom of Infinite. Then x^+, x^{++}, x^{+++}, ... are also inductive sets.

Outline

The Construction of the Set of the Natural Numbers

Defining the natural numbers

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The Construction of the Set of the Natural Numbers

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For defining the natural numbers in ZFC we need to define zero and successor following the Dedekind–Peano axioms.

Definition (p. 38)

$\mathbf{0} := \emptyset$.

Definition

A **numeral** $\mathbf{n}$ is the representation of a natural number n in a formal system.

The Construction of the Set of the Natural Numbers

Example

Some numerals from zero and successor.

$$\begin{aligned}
\mathbf{1} &:= \mathbf{0}^+ \doteq \mathbf{0} \cup \{\mathbf{0}\} && \doteq \{\mathbf{0}\},\\
\mathbf{2} &:= \mathbf{1}^+ \doteq \mathbf{1} \cup \{\mathbf{1}\} \doteq \{\mathbf{0}\} \cup \{\mathbf{1}\} && \doteq \{\mathbf{0}, \mathbf{1}\},\\
\mathbf{3} &:= \mathbf{2}^+ \doteq \mathbf{2} \cup \{\mathbf{2}\} \doteq \{\mathbf{0}, \mathbf{1}\} \cup \{\mathbf{2}\} && \doteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}.
\end{aligned}$$

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\end{aligned}$$

The same numerals from the empty set and successor.

$$\begin{aligned}
\mathbf{1} &\doteq \varnothing^{+} && \doteq \{\,\varnothing\,\},\\
\mathbf{2} &\doteq \varnothing^{++} && \doteq \{\,\varnothing,\{\,\varnothing\,\}\,\},\\
\mathbf{3} &\doteq \varnothing^{+++} && \doteq \{\,\varnothing,\{\,\varnothing\,\},\{\,\varnothing,\{\,\varnothing\,\}\,\}\,\}.
\end{aligned}$$

The Construction of the Set of the Natural Numbers

Definition (p. 40)

Let y be any inductive set. The **set of the natural numbers**, denoted $\mathbb{N}$, is the intersection of all inductive subsets of y, that is,

$$\mathbb{N} := \bigcap \{ z : z \text{ is an inductive subset of } y \}, \qquad \text{(naive definition)},$$

$$\mathbb{N} := \bigcap \{ z \in \mathscr{P}(y) : z \text{ is inductive} \}, \qquad \text{(ZFC definition)}.$$

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Definition (p. 40)

A **natural number** is a member of $\mathbb{N}$.

Outline

Axiom of Replacement

$\forall x \exists y \forall y' \, (y' \in y \leftrightarrow \exists x' \, (x' \in x \wedge \phi(x', y')))$,

where $\phi(s, t)$ is a formula (possibly referring to named sets (possibly with parameters)) such that $\forall s \exists t \, (\phi(s, t) \wedge \forall t' \, (\phi(s, t') \to t' \doteq t))$.

The Axiom of Replacement

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Informal description

"*If $\phi(s, t)$ is a class function, then when its domain is restricted to [a set] x, the resulting images form a set y.*" *(p. 92)*

Category

Axiom for building a set from other set.

The Axiom of Replacement

Example

From a formula (class function) $\phi(s, t)$ we can define a proper class.

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For example, let $\phi(s,t)\colon\ s \doteq t$ be a formula (class function). From $\phi(s,t)$ we can define the proper class

$$\mathscr{A} := \{\, \langle s,s\rangle : s \text{ is a set and } \phi(s,s) \text{ holds} \,\}.$$

The Axiom of Replacement

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$$\mathscr{A} := \{\, \langle s, s\rangle : s \text{ is a set and } \phi(s,s) \text{ holds} \,\}.$$

Note that $\mathscr{A}$ is a proper class because the universe of sets $\mathscr{V}$ is the proper class of first coordinates of the ordered pairs it contains.

The Axiom of Replacement

Exercise 4.23

Let x be a set. Show that $\{\{y\} : y \in x\}$ is a set.

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Solution

Let $\phi(s,t)\colon\ t \doteq \{s\}$ be a formula (class function).

By the Axiom of Replacement when the formula (class function) $\phi(s,t)$ is restricted to the set x there is a set formed with the images of the class function, that is, $\{\{y\} : y \in x\}$ is a set.

The Axiom of Replacement

Remark

Without the Axiom of Replacement we could not prove the existence of the set

$$\{\mathbb{N}, \mathscr{P}(\mathbb{N}), \mathscr{P}(\mathscr{P}(\mathbb{N})), \mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N}))), \dots\}.$$

The Axiom of Replacement

Technical remark

Since there is an instance of the Axiom of Replacement for each formula (class function) $\phi(s,t)$ such that $\forall s \exists t\,(\phi(s,t) \wedge \forall t'\,(\phi(s,t') \rightarrow t' \doteq t))$, some authors define it as an axiomatic schema.†

†See, e.g., (Enderton 1977; Hrbacek and Jech [1978] 1999; Button 2021).

Outline

The Axiom of Foundation

Axiom of Foundation

$\forall x\,(x \mathrel{\dot{\neq}} \emptyset \;\rightarrow\; \exists y\,(y \in x \;\wedge\; x \cap y \doteq \emptyset))$.

The Axiom of Foundation

Axiom of Foundation

$\forall x\,(x \not\doteq \emptyset \;\rightarrow\; \exists y\,(y \in x \;\wedge\; x \cap y \doteq \emptyset))$.

Category

Axiom determining a property of sets.

The Axiom of Foundation

Axiom of Foundation

$\forall x\,(x \not\doteq \varnothing \;\rightarrow\; \exists y\,(y \in x \;\wedge\; x \cap y \doteq \varnothing))$.[†]

Category

Axiom determining a property of sets.

[†] See the corrections to the textbook.

The Axiom of Foundation

Exercise 4.41(a)

Let z by a set. Show that $z \notin z$.

The Axiom of Foundation

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Solution

The singleton $\{z\}$ is a set by the Axiom of Pairs whose unique element is z. By the Axiom of Foundation

$$\{z\} \cap z \doteq \emptyset.$$

Therefore, $z \notin z$ (otherwise $\{z\} \cap z \not\doteq \emptyset$).

The Axiom of Foundation

Definition

Let $\langle x_n \rangle$ be a sequence. An **infinite descending $\in$-chain** is an sequence such that $x_{n+1} \in x_n$, for all $n \in \mathbb{N}$, that is,

$$\cdots x_{n+1} \in x_n \in \cdots \in x_2 \in x_1 \in x_0.$$

The Axiom of Foundation

Theorem 4.3 (No infinite descending $\in$-chains)

Let $\langle x_n \rangle$ be a sequence. The sequence cannot form an infinite descending $\in$-chain.

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Let $\langle x_n \rangle$ be a sequence. The sequence cannot form an infinite descending $\in$-chain.

Proof (by contradiction)

Suppose that $\langle x_n \rangle$ forms an infinite descending $\in$-chain.

Let $x = \{ x_n : n \in \mathbb{N} \}$. By the Axiom of Foundation exists y such that $x \cap y \doteq \emptyset$.

Since $y \in x$, then $y = x_n$ for some n. Since $x_{n+1} \in x$ and $x_{n+1} \in y$ then $x \cap y \not\doteq \emptyset$.

Contradiction!

Outline

References

Tim Button (2021). Set Theory. An Open Introduction. Fall 2021. Revision 2138a50 (2026-07-11). Independently published. URL: https://builds.openlogicproject.org/courses/set-theory/ (cit. on p. 34).

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Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 34).