MT8002 Elements of Set Theory
The Zermelo–Fraenkel Axioms
Axioms 4 to 6

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Conventions

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

See the appendix in the course homepage for the logical symbols, mathematical symbols and acronyms not defined in these slides.

Outline

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Introduction

Formal language

Recall that in ZFC the concepts of pure set and membership relation are primitive (undefined). Also recall that the universe of discourse of the theory is the pure sets.

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Classification

- Axioms stating the existence of sets
- Axioms determining properties of sets
- Axioms for building sets from other sets

Outline

The Axiom of Separation

Axiom of Separation

$\forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \phi(z)))$,

where $\phi(z)$ is a formula (possibly referring to named sets (possibly with parameters)) with free variable z.

The Axiom of Separation

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$\forall x \exists y \forall z (z \in y \ \leftrightarrow \ (z \in x \ \land \ \phi(z)))$,

where $\phi(z)$ is a formula (possibly referring to named sets (possibly with parameters)) with free variable z.

Informal description

"*For any set* x *there is a set consisting of all* z *in* x *for which* $\phi(z)$ *holds.*" *(p. 82)*

Category

Axiom for building a set from other set.

The Axiom of Separation

Example

Let $\phi(x)\colon\ x \notin x$. The Axiom of Separation postulates that

for any set x, there is a set y such that for any set z, $z \in y$ if and only if $z \in x$ and $x \notin x$ (the set y is the empty set).

The Axiom of Separation

Example

Let A and B two given sets and let

$\phi(z, B) \colon z \in B$

the formula with free variable z referring to the named set B (with parameter B).

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Let A and B two given sets and let

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For using the Axiom of Separation on A, B and $\phi(z, B)$ we prove that there is a set y such that for any set z, $z \in y$ if and only if $z \in A$ and $z \in B$ (the set y is the intersection of A and B).

The Axiom of Separation

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Let A and B two given sets and let

$\phi(z, B)\colon\ z \in B$

the formula with free variable z referring to the named set B (with parameter B).

For using the Axiom of Separation on A, B and $\phi(z, B)$ we prove that there is a set y such that for any set z, $z \in y$ if and only if $z \in A$ and $z \in B$ (the set y is the intersection of A and B).

Question

Can you define the union of A and B using something similar to the previous example?

The Axiom of Separation

Theorem

The set postulated by the Axiom of Separation is unique.

The Axiom of Separation

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Proof

The Axiom of Separation states that for any set x, there is one set y, such that for any set z, $z \in y$ if and only if $z \in x$ and $\phi(z)$.

Let y' be other set such that for any set z, $z \in y'$ if and only if $z \in x$ and $\phi(z)$. Then $z \in y$ if and only if $z \in y'$. Therefore, $y \doteq y'$ by the Axiom of Extensionality.

The Axiom of Separation

Definition (p. 82)

Given the existence and uniqueness of the set postulated by the Axiom of Separation we can give the following definition:

Let x be a set. For each propositional function $\phi(z)$ (possibly referring to named sets (possibly with parameters (possibly with parameters)) we denoted by

$$\{ z \in x : \phi(z) \}$$

the set y in the Axiom of Separation such that

$$\forall z (z \in y \leftrightarrow (z \in x \wedge \phi(z))).$$

The Axiom of Separation

Remark

Note that the Axiom of Separation is a restriction of the Naive Comprehension Principle.

$$\{z : \phi(z)\} \qquad \text{(Naive Comprehension Principle)},$$
$$\{z \in x : \phi(z)\} \qquad \text{(Axiom of Separation)}.$$

"*The [Axiom of Separation] is thus saying that, given a set x, there are certain sets definable as subsets of x.*" *(p. 83)*

The Axiom of Separation

Historical remark

Zermelo ([1908] 1967, p. 202) stated the Axiom of Separation using a set and a "definite property", where the elements of the set that satisfy the definite property form a set. However, the concept of "definite property" was not accurate enough. Fraenkel ([1922] 1967) reformulated the Axiom of Separation using other concept. Skolem ([1922] 1967, pp. 202–203) defined the concept of "definite property" as it is used today, that is, it is propositional function in the formal language of ZFC. A definite similar to Skolem's was proposed by Weyl ([1910] 2018).†

†See the introductory note by Jean van Heijenoort to (Fraenkel [1922] 1967) and (Feferman 2000).

The Axiom of Separation

Technical remark

Since there is an instance of the Axiom of Separation for each propositional function $\phi(x)$ some authors define it as an axiomatic schema.[†]

[†] See, e.g., (Enderton 1977; Hrbacek and Jech [1978] 1999; Button 2021).

Outline

Subsets

Definition

A set x is a **subset** of a set y, denoted $x \subseteq y$, if every element of x belongs to y, that is,

$$x \subseteq y := \forall w \, (w \in x \ \rightarrow \ w \in y).$$

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Definition

Let x be a subset y. The set x is a **proper subset** of y if $x \neq y$.

Outline

The Axiom of Power Set

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$\forall x \exists y \forall z (z \in y \;\leftrightarrow\; z \subseteq x)$.

Informal description

"*For any set x there is a set consisting of all subsets of x.*" *(p. 82)*

Category

Axiom for building a set from other sets.

Remark

"*Whether [this axiom] represents an intuitive statement about infinite sets is debatable. But as it leads to some of the most exciting parts of Cantor's original work on infinite numbers, there is not much point in arguing about its inclusion!*" *(p. 83)*

The Axiom of Power Set

Theorem

The set postulated by the Axiom of Power Set is unique.

The Axiom of Power Set

Definition (p. 82)

Given the existence and uniqueness of the set postulated by the Axiom of Power Set we can give the following definition:

Let x be a set. The **power set** of x, denoted $\mathscr{P}(x)$, is the set y in the Axiom of Power Set such that

$$\forall z (z \in y \leftrightarrow z \subseteq x).$$

That is, $\mathscr{P}(x)$ is the set of all subsets of x.

Outline

The Axiom of Union

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$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x))$.

Informal description

"*For any set* x *there is a set which is the union of all the elements of* x." *(p. 82)*

Category

Axiom for building a set from other sets.

The Axiom of Union

Theorem

The set postulated by the Axiom of Union is unique.

The Axiom of Union

Definition (p. 82)

Given the existence and uniqueness of the set postulated by the Axiom of Union we can give the following definition:

Let x be a set. The (**generalised**) **union** of x, denoted $\bigcup x$, is the set y in the Axiom of Union such as

$$\forall z\,(z \in y \;\leftrightarrow\; \exists w\,(z \in w \,\land\, w \in x)).$$

The Axiom of Union

Example

(i) Let x and y be sets. Then, $z \in \bigcup\{x, y\}$ if and only if $z \in x$ or $z \in y$ (see Exercise 4.16).

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(ii) $\bigcup \varnothing \doteq \varnothing$.

Definition (p. 83)

Let x, y be sets. The **union** of x and y, denoted $x \cup y$, is the set $\bigcup\{x, y\}$, that is,

$$x \cup y := \bigcup\{x, y\}.$$

The Axiom of Union

Exercise

Let x_1, x_2, x_3 be sets. Show that there is a set y such that $z \in y$ if and only if $z \doteq x_1$ or $z \doteq x_2$ or $z \doteq x_3$.

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Solution

By the Axiom of Pairs $\{x_1, x_2\}$ and $\{x_3\}$ are sets. Now, by the definition of union of sets and the Axiom of Union, we can define the set y by

$$\{x_1, x_2\} \cup \{x_3\} \doteq \bigcup \{\{x_1, x_2\}, \{x_3\}\}.$$

The Axiom of Union

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Notation

From the previous exercise we extended the curly bracket notation denoting by $\{x_1, x_2, x_3\}$ the set whose elements are x_1, x_2, x_3. In general, $\{x_1, x_2, \dots, x_n\}$ denotes the set with $x_1, x_2, \dots, x_n$ as its elements.

Outline

Intersection of Sets

Definition (p. 85)

Let x and y be sets. The **intersection** of x and y, denoted $x \cap y$, is the set defined by

$$x \cap y := \{ z : (z \in x \ \wedge \ z \in y) \}, \qquad \text{(naive definition)},$$

Intersection of Sets

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Let x and y be sets. The **intersection** of x and y, denoted $x \cap y$, is the set defined by

$$x \cap y := \{ z : (z \in x \ \wedge \ z \in y) \}, \qquad \text{(naive definition)},$$

$$x \cap y := \{ z \in x \cup y : (z \in x \ \wedge \ z \in y) \}, \qquad \text{(ZFC definition)}.$$

Intersection of Sets

Definition (Exercise 4.22)

Let x be a set. The (**generalised**) **intersection** of x, denoted $\bigcap x$, is the set defined by

$$\bigcap x := \{z : z \in y \text{ for all } y \in x\}, \qquad \text{(naive definition)},$$

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$$\bigcap x := \{ z : z \in y \text{ for all } y \in x \}, \qquad \text{(naive definition)},$$

$$\bigcap x := \{ z \in \bigcup x : z \in y \text{ for all } y \in x \}, \qquad \text{(ZFC definition)}.$$

Outline

Cartesian Product

Definition (p. 86)

Let X and Y be sets. The **Cartesian Product** of X and Y, denoted $X \times Y$, is the set defined by

$$X \times Y := \{\langle x, y\rangle : (x \in X \wedge y \in Y)\}, \qquad \text{(naive definition)},$$

(continued on next slide)

Cartesian Product

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$$X \times Y := \{\langle x, y\rangle : (x \in X \wedge y \in Y)\}, \qquad \text{(naive definition)},$$

$$X \times Y := \{z : \exists x \exists y\, ((x \in X \wedge y \in Y) \wedge z \doteq \langle x, y\rangle)\}, \qquad \text{(naive definition)},$$

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$$X \times Y := \{ z \in \mathscr{P}(\mathscr{P}(X \cup Y)) : \exists x \exists y ((x \in X \wedge y \in Y) \wedge z \doteq \langle x, y \rangle) \}, \qquad \text{(ZFC definition)}.$$

(continued on next slide)

Cartesian Product

Definition (p. 86, continuation)

Note that $z \doteq \{\{x\}, \{x, y\}\}$.

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Since $x \in X$ and $y \in Y$ then $x, y \in X \cup Y$.

Cartesian Product

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Note that $z \doteq \{\{x\},\{x,y\}\}$.

Since $x \in X$ and $y \in Y$ then $x, y \in X \cup Y$.

Also note that $\{x,y\} \in \mathscr{P}(X \cup Y)$ and $\{x\} \in \mathscr{P}(X \cup Y)$.

Cartesian Product

Definition (p. 86, continuation)

Note that $z \doteq \{\{x\},\{x,y\}\}$.

Since $x \in X$ and $y \in Y$ then $x, y \in X \cup Y$.

Also note that $\{x,y\} \in \mathscr{P}(X \cup Y)$ and $\{x\} \in \mathscr{P}(X \cup Y)$.

Therefore $\{\{x\},\{x,y\}\} \in \mathscr{P}(\mathscr{P}(X \cup Y))$.

Outline

Functions

Definitions (informal)

Let A and B be sets and let $f : A \to B$ be a function.

(i) The **graph** of f is the set of ordered pairs $\{(a, f(a) : a \in A\}$ (p. 87).

(ii) The **domain** of f is the set A and the **codomain** of f is the set B.

Functions

Definitions (p. 87)

Let A, B and F be sets with $F \subseteq A \times B$. The set F is a **function** with **domain** A and **codomain** B, denoted $F\colon A \to B$, if

$$\forall a\,(a \in A \;\rightarrow\; \exists b\,(b \in B \wedge \langle a, b\rangle \in F \wedge \forall b'\,(\langle a, b'\rangle \in F \;\rightarrow\; b \doteq b'))).$$

Functions

Definitions (p. 88)

(i) Let F be a set of ordered pairs. The set F is a **function** if

$$\forall a \forall b \forall b' \left(\left(\langle a, b \rangle \in F \wedge \langle a, b' \rangle \in F \right) \rightarrow b \doteq b' \right)).$$

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(ii) The **domain** of a function F, denoted $\mathrm{dom}(F)$, is the set defined by

$$\mathrm{dom}(F) := \{ a : \exists b \, \langle a, b \rangle \in F \}, \qquad \text{(naive definition)},$$

$$\mathrm{dom}(F) := \left\{ a \in \bigcup \left(\bigcup F \right) : \exists b \, \langle a, b \rangle \in F \right\}, \qquad \text{(ZFC definition)}.$$

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(iii) The **range** of a function F, denoted $\mathrm{range}(F)$, is the set defined by

$$\mathrm{range}(F) := \left\{ b \in \bigcup \left(\bigcup F \right) : \exists a \, \langle a, b \rangle \in F \right\}, \qquad \text{(naive definition)},$$

$$\mathrm{range}(F) := \left\{ b \in \bigcup \left(\bigcup F \right) : \exists a \, \langle a, b \rangle \in F \right\}, \qquad \text{(ZFC definition)}.$$

Functions

Definition (p. 89)

Let X and Y be sets. The **set of all functions** from Y to X, denoted X^Y, is defined by

$$X^Y := \{ F : F \text{ is a function from } Y \text{ to } X \}, \qquad \text{(naive definition)},$$

$$X^Y := \{ F \in \mathscr{P}(Y \times X) : F \text{ is a function} \}, \qquad \text{(ZFC definition)}.$$

Functions

Definition (p. 90)

Let C and S be sets. The set S is a **sequence of elements of the set** C, denoted $\langle S(n) \rangle$, if S is a function with domain $\mathbb{N}$ and codomain C.

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Remark

The set of the natural numbers $\mathbb{N}$ will be defined in ZFC later.

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Remark

The set of the natural numbers $\mathbb{N}$ will be defined in ZFC later.

Notation

A sequence $\langle S(n)\rangle$ will be denoted by $\langle x_n\rangle$.

Outline

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