MT8002 Elements of Set Theory
The Zermelo–Fraenkel Axioms
Axioms 1 to 3

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Conventions

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

See the appendix in the course homepage for the logical symbols, mathematical symbols and acronyms not defined in these slides.

Outline

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Introduction

Formal language

Recall that in ZFC the concepts of pure set and membership relation are primitive (undefined). Also recall that the universe of discourse of the theory is the pure sets.

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Classification

- Axioms stating the existence of sets
- Axioms determining properties of sets
- Axioms for building sets from other sets

Introduction

Notation

$x \not\doteq y := \neg\, x \doteq y$,

$x \notin y := \neg\, x \in y$.

Outline

The Axiom of Extensionality

Axiom of Extensionality

$\forall x \forall y \, (x \doteq y \;\leftrightarrow\; \forall z \, (z \in x \;\leftrightarrow\; z \in y))$.

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$\forall x \forall y \, (x \doteq y \;\leftrightarrow\; \forall z \, (z \in x \;\leftrightarrow\; z \in y))$.

Informal description

"*Two sets are equal if and only if they contain the same elements.*" *(p. 76)*

Category

Axiom determining a property of sets.

Remark

"*The axiom of extensionality is a statement of the very standard connection between set equality and set membership.*" *(p. 76)*

The Axiom of Extensionality

Theorem

There exists only one set with no elements.

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Proof

Suppose that x and y are both sets without elements. Then $x \doteq y$ by the Axiom of Extensionality. That is, there is exactly one set with no elements.

The Axiom of Extensionality

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There exists only one set with no elements.

Proof

Suppose that x and y are both sets without elements. Then $x \doteq y$ by the Axiom of Extensionality. That is, there is exactly one set with no elements.

Question

Why the above proof is wrong?

The Axiom of Extensionality

Theorem

The identity (equality) relation is compatible with the membership relation.

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The identity (equality) relation is compatible with the membership relation.

That is, for all x, y and z,

(i) if $x \doteq y$ and $x \in z$, then $y \in z$,

(ii) if $x \doteq y$ and $z \in x$, then $z \in y$.

The Axiom of Extensionality

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(i) if $x \doteq y$ and $x \in z$, then $y \in z$,

(ii) if $x \doteq y$ and $z \in x$, then $z \in y$.

Or, equivalently,

$$\frac{x \doteq y \qquad x \in z}{y \in z}, \qquad\qquad \frac{x \doteq y \qquad z \in x}{z \in y}.$$

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Or, equivalently,

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Proof

(i) Hint. To use that the identity is substitutive and the property $\phi(x, y)\colon x \in y$.

(ii) Hint. To use the Axiom of Extensionality.

The Axiom of Extensionality

Technical remark

From the definition of the bi-conditional we can split the Axiom of Extensionality in the following formulae:

$$\forall x \forall y\,(x \doteq y \;\rightarrow\; \forall z\,(z \in x \;\leftrightarrow\; z \in y)), \tag{1}$$

$$\forall x \forall y\,(\forall z\,(z \in x \;\leftrightarrow\; z \in y) \;\rightarrow\; x \doteq y). \tag{2}$$

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$$\forall x \forall y\, (\forall z\, (z \in x \;\leftrightarrow\; z \in y) \;\rightarrow\; x \doteq y). \tag{2}$$

Because (1) is a theorem in FOLEQ some authors state the axiom using only (2).[†]

[†] See, e.g., (Enderton 1977; Hrbacek and Jech [1978] 1999; Button 2021).

The Axiom of Extensionality

Technical remark

ZFC could be defined on FOL instead of FOLEQ, as mentioned in the textbook. Equality could be defined by:[†]

$$x \doteq y := \forall z (z \in x \leftrightarrow z \in y).$$

[†]See, e.g., (Takeuti and Zaring [1971] 1982, Def. 3.1).

Outline

The Axiom of Empty Set

Axiom of Empty Set

$\exists x \forall y \; y \notin x$.

The Axiom of Empty Set

Axiom of Empty Set

$\exists x \forall y \; y \notin x$.

Informal description

"*There is a set with no elements.*" *(p. 76)*

Category

Axiom stating the existence of set.

The Axiom of Empty Set

Question

How many sets postulates the Axiom of Empty Set?

The Axiom of Empty Set

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How many sets postulates the Axiom of Empty Set?

Question

Why the Axiom of Empty Set is not stated by

$\exists! x \forall y \; y \notin x$?

The Axiom of Empty Set

Technical remark

Strictly speaking, the universe of the discourse of FOL (and FOLEQ and ZFC) is not empty. For example, let $\phi(x)$ be a property, then $\forall x \phi(x) \rightarrow \exists x \phi(x)$ is a FOL theorem.[†]

[†] See, e.g., (Lambert 2001).

Outline

The Axiom of Pairs

Axioms of Pairs

$\forall x \forall y \exists z \forall w \, (w \in z \leftrightarrow (w \doteq x \lor w \doteq y))$.

Axioms of Pairs

$\forall x \forall y \exists z \forall w \, (w \in z \leftrightarrow (w \doteq x \vee w \doteq y))$.

Informal descriptions

- "*For any two sets, there is a set whose elements are precisely these sets.*" *(p. 76)*
- "*For any sets* x *and* y*, there is a set having as members just* x *and* y*.*" *(Enderton 1977)*

Category

Axiom for building a set from other sets.

The Axiom of Pairs

Exercise

To prove that the set postulated by the Axiom of Pair is unique (see Exercise 4.8).

Outline

Definitions from Axioms 1 to 3

Definitions (p. 78)

Given the existence and uniqueness of the sets postulated by the previous axioms we can give the following definitions:

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The set postulated by the Axiom of Empty Set is the **empty set**. This set is denoted by $\emptyset$.

Definitions from Axioms 1 to 3

Definitions (p. 78)

Given the existence and uniqueness of the sets postulated by the previous axioms we can give the following definitions:

The set postulated by the Axiom of Empty Set is the **empty set**. This set is denoted by $\emptyset$.

Let x and y be sets. The set postulated by the Axiom of Pairs is the **unordered pair** of x and y. This set is denoted by $\{x, y\}$.

Definitions from Axioms 1 to 3

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Given the existence and uniqueness of the sets postulated by the previous axioms we can give the following definitions:

The set postulated by the Axiom of Empty Set is the **empty set**. This set is denoted by $\varnothing$.

Let x and y be sets. The set postulated by the Axiom of Pairs is the **unordered pair** of x and y. This set is denoted by $\{x, y\}$.

Remark

Note that we introduced the familiar notation of curly brackets $\{\ \}$ only for denoting the unordered pair.

Definitions from Axioms 1 to 3

Exercise

To prove that $\{x, x\} \doteq \{x\}$ (see Exercise 4.9).

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Theorem 4.1

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Exercise

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Theorem 4.1

For any set x there is a set whose only element is x.

Proof

Let x be a set. By the Axiom of Pairs $\{x, x\}$ is a set. By the Axiom of Extensionality the set $\{x, x\}$ is equal to the set $\{x\}$. The set $\{x\}$ is a set whose only element is x.

Definitions from Axioms 1 to 3

Definition (p. 78)

Let x be a set. The **singleton** of x, denoted $\{ x \}$, is the set that has x as its only element.

Outline

Ordered Pairs

Definition (p. 79)

Let x and y be sets. We define the **ordered pair** of x and y, denoted $\langle x, y\rangle$, where x is the **first coordinate** and y is the **second coordinate**, using Kuratowski (1921)'s definition, that is,

$$\langle x, y\rangle := \{\{x\},\{x, y\}\}.$$

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Question

Is $\langle x, y\rangle$ a set?

Ordered Pairs

Theorem

Let x and y be sets. The ordered pair $\langle x, y \rangle$ is a set.

Ordered Pairs

Theorem

Let x and y be sets. The ordered pair $\langle x, y \rangle$ is a set.

Proof

Let x and y be sets. By the Axiom of Pairs $\{ x, y \}$ and $\{ \{ x \}, \{ x, y \} \}$ are sets. Therefore, $\langle x, y \rangle$ is a set.

Ordered Pairs

Theorem

Let x and y be sets. The set $\langle x, y \rangle$ is unique.

Ordered Pairs

Theorem

Let x and y be sets. The set $\langle x, y \rangle$ is unique.

Proof

Let x and y be sets. By the uniqueness of the set postulated by Axiom of Pairs the set $\langle x, y \rangle$ is unique.

Ordered Pairs

Example

We show that $\langle \emptyset, \{\emptyset\}\rangle \neq \langle \{\emptyset\}, \emptyset\rangle$.

$$\langle \emptyset, \{\emptyset\}\rangle := \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \neq \{\{\{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\} =: \langle \{\emptyset\}, \emptyset\rangle.$$

Ordered Pairs

Example

Let x be a set. Then

$$\begin{aligned} \langle x, x\rangle &:= \{\{x\},\{x,x\}\} \\ &\doteq \{\{x\},\{x\}\} \\ &\doteq \{\{x\}\}. \end{aligned}$$

Ordered Pairs

Theorem 4.2 (The ordered pair property)

Let x, y, u and v be sets. If $\langle x, y\rangle \doteq \langle u, v\rangle$, then $x \doteq u$ and $y \doteq v$.

Outline

Ordered Tuples

Definition (p. 80)

Let $x_1, x_2, x_3, \dots, x_n$ be sets, with $n \geq 3$. The **ordered n-tuple**, for $n \geq 3$, is recursively (inductively) defined by

$$\langle x_1, x_2, x_3, \dots, x_n \rangle := \langle x_1, \langle x_2, x_3, \dots, x_n \rangle \rangle.$$

Ordered Tuples

Example

Let x_1, x_2, x_3 and x_4 be sets. The **ordered triple** $\langle x_1, x_2, x_3 \rangle$ and the **ordered quadruple** $\langle x_1, x_2, x_3, x_4 \rangle$ are recursively (inductively) defined by

$$\langle x_1, x_2, x_3 \rangle := \langle x_1, \langle x_2, x_3 \rangle \rangle,$$
$$\langle x_1, x_2, x_3, x_4 \rangle := \langle x_1, \langle x_2, x_3, x_4 \rangle \rangle.$$

Ordered Tuples

Remark

Let x_1, x_2 and x_3 be sets. Although we can show that $\langle x_1, x_2, x_3 \rangle$ is a set at the moment we have no enough axioms for showing that $\{ x_1, x_2, x_3 \}$ is also a set.

Outline

References

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