MT8002 Elements of Set Theory
The Zermelo–Fraenkel Axioms
A Formal Language

Andrés Sicard-Ramírez

Universidad EAFIT

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Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Conventions

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

See the appendix in the course homepage for the logical symbols, mathematical symbols and acronyms not defined in these slides.

Outline

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First-Order Logic with Equality

Description

"*In general, sentences containing quantifiers are only true or false relative to some* ***domain of discourse*** *or* ***domain of quantification*** *[or* ***universe of discourse****]. Sometimes the intended domain contains all objects there are. Usually, though, the intended domain is a much more restricted collection of things, say the people in the room, or some particular set of physical objects, or some collection of numbers.*" *(Barker-Plummer, Barwise and Etchemendy [1999] 2011, p. 238)*

First-Order Logic with Equality

Definition

A **proposition** is a sentence that can be assigned a truth value of `true` or `false`.

First-Order Logic with Equality

Definitions

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A **property** is a unary propositional function.

First-Order Logic with Equality

Theorem

The equality (identity) relation is an equivalence relation.

First-Order Logic with Equality

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That is, for all X, Y, Z,

(i) $X \doteq X$ (reflexivity),

(ii) if $X \doteq Y$, then $Y \doteq X$ (symmetry),

(iii) if $X \doteq Y$ and $Y \doteq Z$, then $X \doteq Z$ (transitivity).

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Or, equivalently,

$$\frac{}{X \doteq X,}\ \text{(reflexivity)} \qquad \frac{X \doteq Y}{Y \doteq X,}\ \text{(symmetry)} \qquad \frac{X \doteq Y \qquad Y \doteq Z}{X \doteq Z.}\ \text{(transitivity)}$$

First-Order Logic with Equality

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Or, equivalently,

$$\frac{x \doteq y \qquad \phi(x)}{\phi(y).}$$

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First-order theories are characterised by their language (non-logical symbols), which can include constant symbols, function symbols and relation symbols.

Pure Sets

Definition

A **pure set** is a set such that its members are also sets.

First-Order Language of ZFC

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ZFC is a first-order theory of pure sets.

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ZFC have two primitive (undefined) concepts: pure set and membership relation.

The universe of discourse of ZFC is the pure sets.

The language (non-logical symbols) L of ZFC is defined by

$$L = \{\in\},$$

where $\in$ denotes the (binary) membership relation.

First-Order Language of ZFC

Convention

The variables representing arbitrary sets will be denote by lowercase letters $(x, y, z, \ldots)$ and uppercase letters $(X, Y, Z, \ldots)$.

Lowercase and uppercase letters are chosen solely for the sake of readability.

First-Order Language of ZFC

Definition

The **formulae** (or **statements**) of ZFC are inductively defined by:

(i) $x \doteq y$ is a (atomic) formula,

(ii) $x \in y$ is a (atomic) formula,

(iii) if ϕ is a formula then $\neg\phi$ is a formula,

(iv) if ϕ and ψ are formulae then $(\phi \star \psi)$ is a formula, where $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,

(v) if ϕ is a formula then $\forall x \phi$ is a formula,

(vi) if ϕ is a formula then $\exists x \phi$ is a formula.

First-Order Language of ZFC

Example

$\neg\exists x \forall y\ y \in x$

There is no set x such that all sets y are elements of x

First-Order Language of ZFC

Examples

(i) $\exists x \forall y\,(y \in x \ \leftrightarrow\ (y \in A \wedge y \in B))$
Let A and B two given sets. The intersection of A and B is a set.

"*Notice that we are extending our use of variable symbols by permitting such symbols, in this case A and B, to stand as **names** for given sets. In general, if some of the free variables in a formula ϕ are replaced by names, we shall describe the formula as **referring to named sets** and only count the remaining variables as its free variables.*" *(p. 74)*

First-Order Language of ZFC

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(ii) $\forall u \forall v \exists x \forall y\,(y \in x \leftrightarrow (y \in u \wedge y \in v))$
For any pair of sets, their intersection is a set.

First-Order Language of ZFC

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(i) $\exists x \forall y\, (y \in x \;\leftrightarrow\; (y \in A \;\wedge\; y \in B))$
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The term "formula as referring to named sets" usually is called "formula with parameters" in the literature.[†]

(ii) $\forall u \forall v \exists x \forall y\, (y \in x \;\leftrightarrow\; (y \in u \;\wedge\; y \in v))$
For any pair of sets, their intersection is a set.

[†] For an explanation of the use of parameters, see, e.g., (Button 2021, § 7.3).

First-Order Language of ZFC

Properties

"*We shall confine ourselves to those properties P of sets y which can be expressed by a formula [(unary propositional function)] $\phi(y)$ within our formal language with a free variable y.*" *(p. 72)*

First-Order Language of ZFC

Example (p. 72)

The property $P(x)$: “The set x is empty” is expressed by the formula with free variable x

$\forall y \neg y \in x$.

That is, the property is expressed by the unary propositional function

$\phi(x) : \forall y \neg y \in x$.

First-Order Language of ZFC

Exercise 4.3.(c)

The formula $\forall x \exists y\, (\phi(x, y) \land \forall y' (\phi(x, y') \rightarrow\ y' \doteq y))$, where $\phi(x, y)$ is a binary propositional function (formula with free variables x and y), represents that for any set x there is exactly one set y for which $\phi(x, y)$ holds.

Exercise 4.3.(c)

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Note that we can use $\phi(x, y)$ for defining a function $f(x)$.

First-Order Language of ZFC

Example (p. 73)

The existence of a set x consisting of all sets y satisfying a property $\phi(x)$ is represented by the formula

$\exists x \forall y \, (y \in x \ \leftrightarrow \ \phi(y))$.

First-Order Language of ZFC

Question

Can we write down a formula expressing a false sentence about sets?

First-Order Language of ZFC

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Can we write down a formula expressing a false sentence about sets?

Question

Can we write down a formula expressing that an “impossible” set exits?

Outline

References

Dave Barker-Plummer, Jon Barwise and John Etchemendy [1999] (2011). Language, Proof and Logic. In collab. with Liu. Albert. In collab. with Michael Murray. In collab. with Emma Pease. 2nd ed. (cit. on p. 5).

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