CM0889 Analysis of Algorithms
Appendix

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Induction

Principle of Mathematical Induction

Let $P(n)$ be a property on natural numbers n, and let a be a fixed natural number.

If

(i) $P(a)$ is true and

(ii) for every natural number $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true,

then

(iii) $P(n)$ is true for all natural numbers $n \geq a$.

Induction

Principle of Strong Induction

Let $P(n)$ be a property on natural numbers n, and let a be a fixed natural number.

If

(i) $P(a)$ is true and

(ii) for every natural number $k \geq a$, if $P(i)$ is true for $a \leq i \leq k$, then $P(k+1)$ is true,

then

(iii) $P(n)$ is true for all natural numbers $n \geq a$.

Remark

Other names: Principle of complete induction and principle of course-of-values induction.

Induction

Theorem

The principle of mathematical induction and the principle of strong induction are equivalents.

Floor and Ceiling Functions

Definition

The **floor** function is defined by

$$\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$$
$$\lfloor x \rfloor := \text{largest integer less than or equal to } x.$$

Example

$$\begin{aligned} \lfloor 42 \rfloor &= 42, \\ \lfloor 5.42 \rfloor &= 5, \\ \lfloor -5.52 \rfloor &= -6. \end{aligned}$$

Floor and Ceiling Functions

Definition

The **ceiling** function is defined by

$$\begin{aligned} \lceil \cdot \rceil &: \mathbb{R} \to \mathbb{Z} \\ \lceil x \rceil &:= \text{smallest integer greater than or equal to } x. \end{aligned}$$

Example

$$\begin{aligned} \lceil 42 \rceil &= 42, \\ \lceil 5.42 \rceil &= 6, \\ \lceil -5.52 \rceil &= -5. \end{aligned}$$

Logarithms

Definition

For any fixed real number $b > 1$,

$$\log_b : \mathbb{R}^+ \to \mathbb{R}$$
$$\log_b x = y \quad \text{iff} \quad b^y = x.$$

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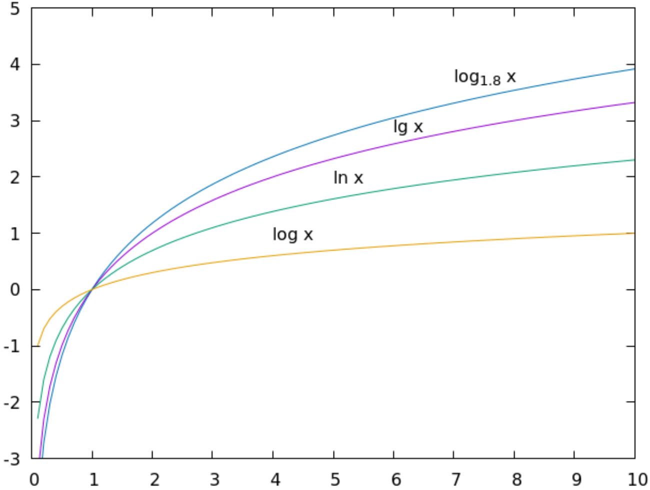
Notation

$\lg x$: Logarithm on base 2
$\ln x$: Logarithm on base e
$\log x$: Logarithm on base 10

Logarithms

Logarithmic functions $\lg x$, $\ln x$, $\log x$ and $\log_{1.8} x$



Logarithms

Properties

For any fixed real number $b > 1$, for all $x, y \in \mathbb{R}^+$, and for all $z \in \mathbb{R}$:

$$\begin{aligned}
\log_b(xy) &= \log_b x + \log_b y, \\
\log_b(x/y) &= \log_b x - \log_b y, \\
\log_b(x^z) &= z \log_b x.
\end{aligned}$$

Logarithms

Properties

For all real numbers a and b greater than 1 and for all $x \in \mathbb{R}^+$:

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Logarithms

Properties

For all real numbers a and b greater than 1, for all $x \in \mathbb{R}^+$, if $a < b$ then

$$\log_a x > \log_b x.$$

Summations

Definition

Let $a_1, a_2, \ldots, a_n$ be a sequence of numbers, where n is a positive integer. Recall the recursive definition of the **summation notation**:

$$\begin{aligned}\sum_{k=1}^{1} a_k &:= a_1,\\ \sum_{k=1}^{n} a_k &:= \left(\sum_{k=1}^{n-1} a_k\right) + a_n\\ &= a_1 + a_2 + \cdots + a_{n-1} + a_n.\end{aligned}$$

Summations

Properties

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \qquad \text{(additive property),}$$

$$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k \qquad \text{(homogeneous property),}$$

$$\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k \qquad \text{(linearity property).}$$

Summations

Properties

$$\sum_{k=1}^{n} f(n) = nf(n),$$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{i} a_k + \sum_{k=i+1}^{n} a_k.$$

Summations

Properties

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$