CM0859 – MT5009 Type Theory
Constructivism

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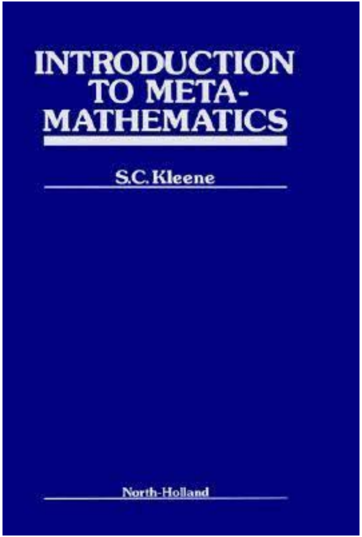
Preliminaries

- "Textbook"
  "Lecture Notes on What is (Constructive) Logic?" (Pfenning 2023).
- Other reference
  Constructivism in Mathematics. An Introduction. Volume I (Troelstra and van Dalen 1988).

The Crisis in the Foundations of Mathematics



The Crisis in the Foundations of Mathematics

$$\text{Paradoxes} \Rightarrow \text{Crisis} \Rightarrow \begin{cases} \text{Logicism} & \text{(Russell and Whitehead)} \\ \text{Formalism} & \text{(Hilbert)} \\ \text{Intuitionism} & \text{(Brouwer)} \end{cases}$$

The Crisis in the Foundations of Mathematics

Logicism (Russell and Whitehead)

"The logicistic thesis is that mathematics is a branch of logic. The mathematical notions are to be defined in terms of the logical notions. The theorems of mathematics are to be proved as theorems of logic." (Kleene [1952] 1974, p. 43)

The Crisis in the Foundations of Mathematics

Formalism (Hilbert)

"Classical mathematics shall be formulated as a formal axiomatic theory, and this theory shall be proved to be consistent, i.e. free from contradiction." (Kleene [1952] 1974, p. 53)

The Crisis in the Foundations of Mathematics

Intuitionism (Brouwer)

> *"Intuitionism is based on the idea that mathematics is a creation of the mind. The truth of a mathematical statement can only be conceived via a mental construction that proves it to be true." (Iemhoff 2024)*

The Crisis in the Foundations of Mathematics

Conceptions of the infinite

(i) Non-Intuitionism

"The infinite is treated as actual or completed or extended or existential. An infinite set is regarded as existing as a completed totality, prior to or independently of any human process of generation or construction, and as though it could be spread out completely for our inspection." (Kleene [1952] 1974, p. 48)

(ii) Intuitionism

"The infinite is treated only as potential or becoming or constructive. The recognition of this distinction, in the case of infinite magnitudes, goesback to Gauss, who in 1831 wrote, 'I protest ... against the use of an infinite magnitude as something completed, which is never permissiblein mathematics.' (Werke VIII p. 216.)" (Kleene [1952] 1974, p. 48)

Constructivism

Some differences with classical logic

(i) Rejection of the principle of exclude middle *(tertium non datur)*.

$$\vdash A \vee \neg A, \text{ for all formula } A.$$

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(i) Rejection of the principle of exclude middle *(tertium non datur)*.

$$\vdash A \vee \neg A, \text{ for all formula } A.$$

(ii) A proof of an existential formula $\exists x.A(x)$ must include a witness t such as $A(t)$ is true.

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Constructivism

Some differences with classical logic

(iii) Rejection of proofs by contradiction

Proof by contradiction
(or *reductio ad absurdum*)

$$\dfrac{\begin{matrix}[\neg A] \\ \vdots \\ \bot\end{matrix}}{A}$$

Proof of negation (Bauer 2017)

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ \bot\end{matrix}}{\neg A}$$

Non-Constructive Proofs

Example

To prove that there are irrational numbers $r, s \in \mathbb{R}$ such that r^s is rational.

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Proof (using the principle of exclude middle)

(whiteboard)

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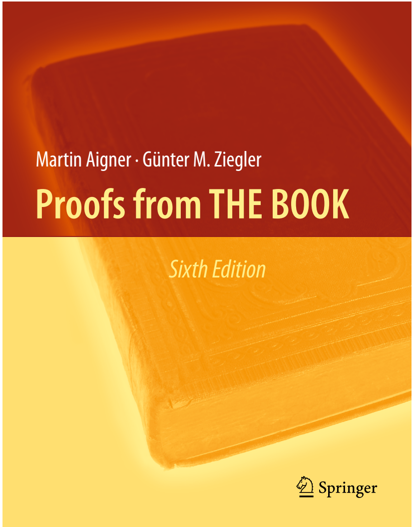
(whiteboard)

Question

Could you give me two irrational numbers r, s such that r^s is rational?

Non-Constructive Proofs



Non-Constructive Proofs

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To prove that there are an infinity number of primes.

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Proof (by contradiction)

> *"**Euclid's proof.** For any finite set $\{p_1, \ldots, p_r\}$ of primes, consider the number $n = p_1 p_2 \cdots p_r + 1$. This n has a prime divisor p. But p is not one of the p_i: otherwise p would be a divisor of n and of the product $p_1 p_2 \cdots p_r$, and thus also of the difference $n - p_1 p_2 \cdots p_r = 1$, which is impossible. So a finite set $\{p_1, \ldots, p_r\}$ cannot be the collection of all prime numbers." (Aigner and Ziegler [1998] 2018, p. 3)*

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Question

Could you give me an infinite list of primes?

Non-Constructive Proofs

Observation

The axiom of choice is a source of non-constructive proofs.

Non-Constructive Proofs

Definition

The **Cartesian product** (or **generalised product**) of a family of sets $\langle A_i \mid i \in I \rangle$ is defined by

$$\bigtimes_{i\in I} A_i := \left\{ f \;\middle|\; f : I \to \bigcup_{i\in I} A_i \text{ and } \forall i\,(i \in I \to f(i) \in A_i) \right\}.$$

Non-Constructive Proofs

Definition

Axiom of choice: Let $\langle H_i \mid i \in I \rangle$ be a family a sets. If $H(i) \neq \emptyset$ for all $i \in I$, then $\bigtimes_{i \in I} H(i) \neq \emptyset$ (Enderton 1977).

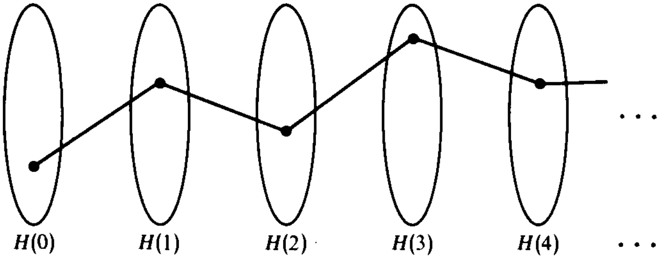

Illustration of the axiom of choice.[†]

[†]Figure source: (Enderton 1977, Fig. 11).

The Brouwer-Heyting-Kolmogorov (BHK) Interpretation

Logical constants

$\wedge$	(and)	conjunction
$\vee$	(or)	(inclusive) disjunction
$\supset$	(if __, then __)	conditional
$\bot$	(falsity)	bottom, falsum
$\forall x$	(for every x)	universal quantifier
$\exists x$	(there exists a x)	existential quantifier

Definition

We define negation by $\neg A := A \supset \bot$.

The Brouwer-Heyting-Kolmogorov (BHK) Interpretation

Constructive interpretation of the logical constants

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- A proof of $\neg A$ is a function (method, program) with transforms a (hypothetical) proof of A into a contradiction.

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- A proof of $\exists x.A$ is a construction of a witness d and a proof of $A(d)$.

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- A proof of $A \supset B$ is a function (method, program) which to each proof of A gives a proof of B.
- There is no proof of $\perp$.
- A proof of $\neg A$ is a function (method, program) with transforms a (hypothetical) proof of A into a contradiction.
- A proof of $\exists x.A$ is a construction of a witness d and a proof of $A(d)$.
- A proof of $\forall x.A$ is a function (method, program) which takes an arbitrary individual d into a proof of $A(d)$.

Connection Between Proofs and Programs

Example

We define the follow predicates on natural numbers:

$$\begin{aligned} \mathrm{even}(x) &:= \exists y. x = 2y, \\ \mathrm{odd}(x) &:= \exists y. x = 2y + 1. \end{aligned}$$

Prove that $\forall x.\mathrm{even}(x) \vee \mathrm{odd}(x)$.

Proof (by induction on x)

(whiteboard)

Connection Between Proofs and Programs

Proof (by induction on x)

(i) **Basis step:** $x = 0$.
Then $\mathrm{even}(x)$ is true because for $y = 0$ (witness), $x = 2y$.

(ii) **Inductive step:** $x = x' + 1$.
For inductive hypothesis $\mathrm{even}(x') \vee \mathrm{odd}(x')$ is true.

- **Case:** $\mathrm{even}(x')$ is true.
  That is, $x' = 2y'$ for some y'. Then $x = 2y' + 1$ and therefore $\mathrm{odd}(x)$ is true and the witness is y'.

- **Case:** $\mathrm{odd}(x')$ is true.
  That is $x' = 2y' + 1$ for some y'. Then $x = (2y' + 1) + 1 = 2(y' + 1)$ and therefore $\mathrm{even}(x)$ is true and the witness is $y' + 1$.

Connection Between Proofs and Programs

Haskell function "from" the proof that $\forall x.\mathrm{even}(x) \vee \mathrm{odd}(x)$

```haskell
data Nat = Zero | Succ Nat

data EO = Even | Odd
  deriving Show

isEvenOrOdd :: Nat -> EO
isEvenOrOdd Zero     = Even
isEvenOrOdd (Succ n) = case isEvenOrOdd n of
  Even -> Odd
  Odd  -> Even
```

Connection Between Proofs and Programs

Haskell function with witness “from” the proof that $\forall x.\text{even}(x) \vee \text{odd}(x)$

```haskell
data Nat = Zero | Succ Nat
  deriving Show

data EO = Even Nat | Odd Nat
  deriving Show

isEvenOrOdd :: Nat -> EO
isEvenOrOdd Zero     = Even Zero
isEvenOrOdd (Succ x) = case isEvenOrOdd x of
  Even y -> Odd y
  Odd y  -> Even $ Succ y
```

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