CM0845 Logic
First-Order Logic: Syntax

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Semester 2016-1

Remark

The references for this section are van Dalen [2013, § 3.1, § 3.2 and § 3.3].

Introduction

Example

Informal examples [van Dalen 2013, p. 53]:

$\exists x P(x)$	there is an x with property P
$\forall y P(y)$	for all y P holds (all y have the property P)
$\forall x \exists y(x = 2y)$	for all x there is a y such that x is two times y
$\forall \epsilon(\epsilon > 0 \to \exists n(n < \epsilon))$	for all positive there is an n such that $n < \epsilon$
$x < y \to \exists z(x < z \land z < y)$	if $x < y$, then there is a z such that $x < z$ and $z < y$
$\forall x \exists y(x \cdot y = 1)$	for each x there exists an inverse y

Structures

Definition

A **structure** is an ordered sequence

$$\langle A, R_1, \ldots, R_n, F_1, \ldots, F_m, \{c_i \mid i \in I\}\rangle,$$

where

(i) A is a non-empty set, the **universe** of the structure,

(ii) $R_1, \ldots, R_n$ are relations on A,

(iii) $F_1, \ldots, F_m$ are functions on A, and

(iv) the c_i, where $i \in I$, are elements of A (constants).

Notation

- Structures are denoted by Gothic capitals: $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$
- $A = |\mathfrak{A}|$

Structures

Examples

Whiteboard.

Structures

Definition

The **similarity type** (or **signature** or **non-logical constants**) of a structure

$$\langle A, R_1, \ldots, R_n, F_1, \ldots, F_m, \{c_i \mid i \in I\}\rangle$$

is a sequence

$$\langle r_1, \ldots, r_n; a_1, \ldots, a_m; \kappa\rangle,$$

where

(i) $R_i \subseteq A^{r_i}$,

(ii) $F_j : A^{a_j} \to A$, and

(iii) $\kappa = |\{c_i \mid i \in I\}|$ (cardinality of I).

Structures

Examples

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Structures

Examples

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Limiting cases

0-ary relations and 0-ary functions.

Structures

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Limiting cases

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Convention

All the structures are equipped implicitly with the identity relation.

Alphabet

Definition

The **alphabet** has the following symbols:

(i) Predicate symbols: $P_1, \ldots, P_n$ and $\doteq$

(ii) Function symbols: $f_1, \ldots, f_m$

(iii) Constant symbols: $\overline{c}_i$ for $i \in I$

(iv) Variables: $x_0, x_1, x_2, \ldots$ (countably many)

(v) Connectives: $\vee, \wedge, \rightarrow, \neg, \leftrightarrow, \bot, \forall, \exists$

(vi) Auxiliary symbols: $(,)$

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(v) Connectives: $\vee, \wedge, \rightarrow, \neg, \leftrightarrow, \perp, \forall, \exists$

(vi) Auxiliary symbols: $(,)$

Remark

The equality symbol.

The Set of Terms

Definition

The **set of terms**, denoted TERM, is the smallest set X with the properties:

(i) $x_i \in X$, where $i \in \mathbb{N}$,

(ii) $\overline{c}_i \in X$, where $i \in I$, and

(iii) $t_1, \ldots, t_{a_i} \in X \Rightarrow f_i(t_1, \ldots, t_{a_i}) \in X$, for $1 \leq i \leq m$.

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Examples

Whiteboard.

The Set of Formulae

Definition

The **set of formulae**, denoted FORM, is the smallest set X with the properties:

(i) $\perp \in X$,

(ii) $P_i \in X$ if $r_i = 0$,

(iii) $t_1, \ldots, t_{r_i} \in \mathrm{TERM} \Rightarrow P_i(t_1, \ldots, t_{r_i}) \in X$,

(iv) $t_1, t_2 \in \mathrm{TERM} \Rightarrow t_1 \doteq t_2 \in X$,

(v) $\varphi, \psi \in X \Rightarrow (\varphi \mathbin{\Box} \psi) \in X$, where $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,

(vi) $\varphi \in X \Rightarrow (\neg\varphi) \in X$,

(vii) $\varphi \in X \Rightarrow ((\forall x_i)\varphi) \in X$ and

(viii) $\varphi \in X \Rightarrow ((\exists x_i)\varphi) \in X$.

The formulae defined in the four first items are called atomic formulae or atoms.

Notational Conventions

- We use the conventions of propositional logic.
- We delete the outer brackets and the brackets round $\forall x$ and $\exists x$ whenever possible.
- Quantifiers bind more strongly than binary connectives.
- Join strings of quantifiers, e.g. $\forall x_1 x_2 \exists x_3 x_4 \varphi$ stands for $\forall x_1 \forall x_2 \exists x_3 \exists x_4 \varphi$.

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Examples

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Induction Principles and Recursive Definitions

Remark

Given that TERM and FORM are set inductively defined, we have induction principles and recursive definitions on them.

Set of Free Variables of a Term

Definition

The **set of free variables of a term t**, denoted $\mathrm{FV}(t)$, is defined by

$$\begin{aligned} \mathrm{FV} : \mathrm{TERM} &\to \{x_i \mid i \in \mathbb{N}\} \\ \mathrm{FV}(x_i) &= \{x_i\}, \\ \mathrm{FV}(\overline{c}_i) &= \emptyset, \\ \mathrm{FV}(f(t_1, \ldots, t_n)) &= \mathrm{FV}(t_1) \cup \cdots \cup \mathrm{FV}(t_n). \end{aligned}$$

Closed Terms

Definition

A term t is **closed** iff $\mathrm{FV}(t) = \emptyset$. The set of closed terms is denoted by TERM_c.

Examples

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Set of Free Variables of a Formula

Definition

The **set of free variables of a formula** φ, denoted $\mathrm{FV}(\varphi)$, is defined by

$$
\begin{aligned}
\mathrm{FV} &: \mathrm{FORM} \to \{x_i \mid i \in \mathbb{N}\} \\
\mathrm{FV}(\bot) &= \emptyset \\
\mathrm{FV}(P) &= \emptyset, \text{ for } P \text{ propositional symbol} \\
\mathrm{FV}(P(t_1, \dots, t_n)) &= \mathrm{FV}(t_1) \cup \dots \cup \mathrm{FV}(t_n), \\
\mathrm{FV}(t_1 \doteq t_2) &= \mathrm{FV}(t_1) \cup \mathrm{FV}(t_2), \\
\mathrm{FV}(\varphi \,\square\, \psi) &= \mathrm{FV}(\varphi) \cup \mathrm{FV}(\psi), \\
\mathrm{FV}(\neg\varphi) &= \mathrm{FV}(\varphi), \\
\mathrm{FV}(\forall x_i \varphi) = \mathrm{FV}(\exists x_i \varphi) &= \mathrm{FV}(\varphi) - \{x_i\}.
\end{aligned}
$$

Sentences

Definition

A formula φ is **closed** iff $\mathrm{FV}(\varphi) = \emptyset$. A closed formula is also called a sentence. The set of sentences is denoted by SENT.

Examples

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Free Terms for a Variable in a Formula

Definition

A **term t is free for a variable x in a formula φ** iff

(i) φ is atomic,

(ii) $\varphi := \neg\psi$ and t is free for x in ψ,

(iii) $\varphi := \varphi_1 \square \varphi_2$ and t is free for x in φ_1 and φ_2,

(iv) $\varphi := \forall y \psi$ and if $x \in \mathrm{FV}(\varphi)$, then $y \notin \mathrm{FV}(t)$ and t is free for x in ψ, or

(v) $\varphi := \exists y \psi$ and if $x \in \mathrm{FV}(\varphi)$, then $y \notin \mathrm{FV}(t)$ and t is free for x in ψ.

The Extended Language

Definition

The **extended language**, $L(\mathfrak{A})$, of $\mathfrak{A}$ is obtained from the language L, of the type of $\mathfrak{A}$, by adding constant symbols for all elements of $|\mathfrak{A}|$. We denote the constant symbol, belonging to $a \in |\mathfrak{A}|$, by $\overline{a}$.

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Examples

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References

van Dalen, Dirk [1980] (2013). Logic and Structure. 5th ed. Springer (cit. on pp. 2, 3).