CM0845 Logic
First-Order Logic: Natural Deduction

Andrés Sicard-Ramírez

Universidad EAFIT

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Remark

The references for this section are van Dalen [2013, § 3.8 and § 3.9].

Derivation Rules for the Quantifiers

Universal quantifier

$$\frac{\varphi(x)}{\forall x \varphi(x)}\,\forall\mathrm{I} \qquad\qquad \frac{\forall x \varphi(x)}{\varphi[\,t/x\,]}\,\forall\mathrm{E}$$

Side condition: In $\forall\mathrm{I}$, the variable x may not occur free in any hypothesis on which $\varphi(x)$ depends.

Derivation Rules for the Quantifiers

Existential quantifier

$$\frac{\varphi[\,t/x\,]}{\exists x \varphi(x)}\ \exists\mathrm{I}$$

$$\frac{\exists x \varphi(x) \qquad \begin{array}{c}[\varphi(x)]^x \\ \vdots \\ \psi\end{array}}{\psi}\ \exists\mathrm{E}^x$$

Side condition: In $\exists\mathrm{E}$, the variable x is not free in ψ, or in a hypothesis of the sub-derivation of ψ, other than $\varphi(x)$.

Derivation Rules for the Quantifiers

Example

The derivation rules for the quantifiers are consistent with the convention that the universe of discourse is not empty, so we can prove that $\vdash \forall x \varphi(x) \to \exists x \varphi(x)$.

Proof

$$\dfrac{\dfrac{\dfrac{[\forall x \varphi(x)]^x}{\varphi(x)}\ \forall\mathrm{E}}{\exists x \varphi(x)}\ \exists\mathrm{I}}{\forall x \varphi(x) \to \exists x \varphi(x)}\ {\to}\mathrm{I}^x$$

Example

Prove that $\vdash \exists x(\varphi(x) \vee \psi(x)) \to \exists x\varphi(x) \vee \exists x\psi(x)$ [van Dalen 2013, p. 92].

Proof

$$
\dfrac{\dfrac{[\exists x(\varphi(x) \vee \psi(x))]^{w} \qquad \dfrac{[\varphi(x) \vee \psi(x)]^{z} \quad \dfrac{\dfrac{[\varphi(x)]^{x}}{\exists x\varphi(x)}\,\exists\mathrm{I}}{\exists x\varphi(x) \vee \exists x\psi(x)}\,\vee\mathrm{I} \quad \dfrac{\dfrac{[\psi(x)]^{y}}{\exists x\psi(x)}\,\exists\mathrm{I}}{\exists x\varphi(x) \vee \exists x\psi(x)}\,\vee\mathrm{I}}{\exists x\varphi(x) \vee \exists x\psi(x)}\,\vee\mathrm{E}^{x,y}}{\exists x\varphi(x) \vee \exists x\psi(x)}\,\exists\mathrm{E}^{z}}{\exists x(\varphi(x) \vee \psi(x)) \to \exists x\varphi(x) \vee \exists x\psi(x)}\,{\to}\mathrm{I}^{w}
$$

Derivation Rules for the Identity

Identity

$$\frac{}{x = x}\,\mathrm{RI}_1 \qquad \frac{x = y}{y = x}\,\mathrm{RI}_2 \qquad \frac{x = y \qquad y = z}{x = z}\,\mathrm{RI}_3$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n}{t[\,x_1, \ldots, x_n/z_1, \ldots, z_n\,] = t[\,y_1, \ldots, y_n/z_1, \ldots, z_n\,]}\,\mathrm{RI}_4$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n \qquad \varphi[\,x_1, \ldots, x_n/z_1, \ldots, z_n\,]}{\varphi[\,y_1, \ldots, y_n/z_1, \ldots, z_n\,]}\,\mathrm{RI}_4$$

Derivation Rules for the Identity

Example

The derivation rules for the identity are consistent with the convention that the universe of discourse is not empty, so we can prove that $\vdash \exists x(x = x)$.

Proof

$$\dfrac{\dfrac{}{x = x}\,\mathrm{RI}_1}{\exists x(x = x)}\,\exists\mathrm{I}$$

References

van Dalen, Dirk [1980] (2013). Logic and Structure. 5th ed. Springer (cit. on pp. 2, 6).