CM0832 Elements of Set Theory
7. Orderings and Ordinals

Andrés Sicard-Ramírez

Universidad EAFIT

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Enderton 1977].

Well-Orderings

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Definition

A **structure** is a pair $\langle A, R\rangle$ consisting of a set A and a binary relation R on A.

Transfinite Induction Principle

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Transfinite induction principle

Let $\langle A, < \rangle$ be a well-ordered structure and assume that B is a subset of A with the special property that for every t in A,

$$\operatorname{seg} t \subseteq B \quad \text{implies} \quad t \in B.$$

Then B coincides with A.

Transfinite Recursion Theorem

Definition

Let $\langle A, < \rangle$ be a well-ordered structure and let B a set. The set of **all functions from initial segments of $\langle A, < \rangle$ into B** is defined by

$$B^{A<} := \{ f \mid f : \operatorname{seg} t \to B, \text{ for some } t \in A \}.$$

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Observation

Let $\langle A, < \rangle$ be a well-ordered structure and let B a set. Note that $B^{A<}$ is a set because we can define it via the subset axiom scheme.

$$B^{A<} := \{ f \in \mathcal{P}(A \times B) \mid f : \operatorname{seg} t \to B, \text{ for some } t \in A \}.$$

Transfinite Recursion Theorem

Transfinite recursion theorem (preliminary form, p. 175)

Let $\langle A, < \rangle$ be a well-ordered structure and let $G : B^{A<} \to B$. Then there is a unique function F such that for any $t \in A$,

$$F : A \to B$$
$$F(t) = G(F \upharpoonright \operatorname{seg} t).$$

Replacement Axiom Scheme

Replacement axiom scheme

For any propositional function $\varphi(x, y)$, not containing B, the following is an axiom:

$$\forall A \, [\, \forall x \, (x \in A \to \exists! y \, \varphi(x, y)) \to \exists B \, \forall y \, (y \in B \leftrightarrow \exists x \, (x \in A \wedge \varphi(x, y))) \,].$$

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The propositional function φ can depend on other variables $t_1, \dots, t_k$. In this case, we use $\varphi(x, y, t_1, \dots, t_k)$ and we universally quantify on variables $t_1, \dots, t_k$ when using the axiom scheme.

Epsilon-Images*

*'The membership symbol ($\in$) is not typographically the letter epsilon but originally it was, and the name "epsilon" persists.' [Enderton 1977, p. 182]

Isomorphisms

Definition

Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be two structures. An **isomorphism** from $\langle A, R\rangle$ onto $\langle B, S\rangle$ is a one-to-one function f from A onto B such that for all $x, y \in A$

$$x \; R \; y \quad \text{iff} \quad f(x) \; S \; f(y).$$

Isomorphisms

Corollary 7H

Let α be the $\in$-image of a well-ordered structure $\langle A, <\rangle$. Then α is a transitive set and $\in_\alpha$ is a well ordering on α, where

$$\in_\alpha := \{\, \langle x, y\rangle \in \alpha \times \alpha \mid x \in y \,\}.$$

Ordinal Numbers

Idea

To assign a ‘number’ to each well-ordered structure that measures its ‘length’. Two well-ordered structures should receive the same number, if and only if, they are isomorphic.

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Theorem 7I

Two well-ordered structures are isomorphic iff they have the same $\in$-image.

Ordinal Numbers

Definition

Let $<$ be a well-ordering on A. The **ordinal number** of $\langle A, <\rangle$ is its ϵ-image. An **ordinal number** is a set that is the ordinal number of some well-ordered structure.

Ordinal Numbers

Definition

A set A is **well-ordered by the membership relation** iff the relation

$$\in_A := \{ \langle x, y \rangle \in A \times A \mid x \in y \}$$

is a well-ordering on A.

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Definition (other definition of ordinal number)

A set A is an **ordinal number** iff [Hrbacek and Jech (1978) 1999, p. 107]:

(i) The set is transitive.

(ii) The set is well-ordered by the membership relation.

Ordinal Numbers

Burali-Forti theorem (p. 194)

There is no set to which every ordinal number belongs.

Well-Ordering Theorem

Well-ordering theorem (p. 196)

For any set A, there is a well-ordering on A

Cardinal Numbers

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Observation

Cardinal numbers and initial ordinals are the same numbers.

Rank

Idea

We want to define hierarchy of sets indexed by ordinals:

$$V_0 = \emptyset,$$
$$V_{\alpha+1} = \mathcal{P}V_\alpha, \text{ if } \alpha \text{ is a succesor ordinal},$$
$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta, \text{ if } \lambda \text{ is a limit ordinal}.$$

Regularity Axiom

Regularity (foundation) axiom

Every non-empty set A has a member m with $m \cap A = \emptyset$, that is,

$$\forall A \, [\, A \neq \emptyset \to \exists m \, (m \in A \wedge m \cap A = \emptyset) \,].$$

References

Enderton, Herbert B. (1977). Elements of Set Theory. Academic Press (cit. on pp. 2, 14).

Hrbacek, Karel and Jech, Thomas [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on pp. 20, 21).