CM0832 Elements of Set Theory
4. Natural Numbers

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Enderton 1977].

Defining the Natural Numbers

Approaches for introducing mathematical objects

- Axiomatic
- Definitional

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- We shall prove the properties of natural numbers from properties of sets.

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- We shall prove the properties of natural numbers from properties of sets.

Question

How to define natural numbers in terms of sets?

Inductive Sets

von Neumann's construction

Informal idea: A natural number is the set of all smaller natural numbers

$$
\begin{aligned}
0 &:= \emptyset, \\
1 &:= \{0\} &&= \{\emptyset\}, \\
2 &:= \{0, 1\} &&= \{\emptyset, \{\emptyset\}\}, \\
3 &:= \{0, 1, 2\} &&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\vdots
\end{aligned}
$$

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\end{aligned}
$$

Some 'extra' properties

$$0 \in 1 \in 2 \in 3 \in \cdots \quad \text{and} \quad 0 \subseteq 1 \subseteq 2 \subseteq 3 \subseteq \cdots .$$

Inductive Sets

A wrong impredicative definition

$$n := \{0, 1, \ldots, n - 1\}.$$

'We cannot just say that a set n is a natural number if its elements are all the smaller natural numbers, because such a "definition" would involve the very concept being defined.' [Hrbacek and Jech (1978) 1999, p. 40]

Inductive Sets

Definition

Let a be a set. The **successor** of a is

$$a^{+} := a \cup \{a\}.$$

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Example

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \emptyset^+, \\ 2 &= \emptyset^{++}, \\ 3 &= \emptyset^{+++}, \\ &\vdots \end{aligned}$$

Inductive Sets

Definition

A set A is **inductive** iff

- $\emptyset \in A$ and
- if $a \in A$ then $a^+ \in A$.

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Question

Are there inductive sets?

Infinite Axiom

Infinity axiom

There exists an inductive set, that is,

$$\exists A \, [\, \emptyset \in A \wedge \forall a \, (a \in A \to a^{+} \in A) \,].$$

The Set of Natural Numbers

Definition

A **natural number** is a set that belongs to every inductive set.

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Theorem 4A

There is a set whose members are exactly the natural numbers.

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Theorem 4A

There is a set whose members are exactly the natural numbers.

Proof.

Let A be an inductive set. By the subset axiom scheme, there is a set

$$\{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

∎

The Set of Natural Numbers

Definition

The set of **all natural numbers**, denoted by ω, is defined by

$$\omega := \{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

That is,

$$x \in \omega \quad \text{iff} \quad x \text{ is a natural number.}$$

The Set of Natural Numbers

Theorem 4B

The set ω is inductive, and it is a subset of every other inductive set.

The Set of Natural Numbers

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The set ω is inductive, and it is a subset of every other inductive set.

Observation

The set w is the smallest inductive set

The Set of Natural Numbers

Observation

Since that the collection of all inductive sets is not a set but a proper class, using class we could define the set of natural numbers by

$$\omega := \bigcap \{ A \mid A \text{ is an inductive set} \}.$$

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Observation

Mendelson [(1973) 2008] in the proof of Theorem ZFC 8 defines the set ω as an intersection of some inductive sets.

Induction Principle for Natural Numbers

Induction principle for ω (p. 69)

Any inductive subset of ω coincides with ω.

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Induction principle for ω (other version) [Hrbacek and Jech (1978) 1999]

Let $P(x)$ be a property. Assume that

(i) $P(0)$ holds,

(ii) for all $n \in \omega$, $P(n)$ implies $P(n^+)$.

Then P holds for all natural numbers n.

Proof.

'This is an immediate consequence of our definition of w. The assumptions i) and ii) simple say that the set $A = \{ n \in \omega \mid P(n) \}$ is inductive. $\omega \subseteq A$ follows.' [Hrbacek and Jech (1978) 1999, p. 42] ∎

Defining Natural Numbers as Sets

Observation

So far, we defined natural numbers on terms of sets. A different point of view is stated by some authors (see, e.g. Benacerraf [1965]).

Induction as Foundations

'Thus inductive definibility is a notion intermediate in strength between predicate and fully impredicative definability. It would be interesting to formulate a coherent conceptual framework that made induction the principal notion. There are suggestions of this in the literature, but the possibility has not yet been fully explored.' [Aczel 1977, p. 780]

Transitive Sets

Definition

Let A be a set. The set A is a **transitive set** iff every member of a member of A is itself a member of A, that is,

$$x \in a \in A \quad \text{implies} \quad x \in A.$$

Transitive Sets

Definition

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Example

Whiteboard.

Transitive Sets

Theorem

A set A is a transitive set iff $\bigcup A \subseteq A$.

(continued on next slide)

Transitive Sets

Proof.

i) (Only if) Let A be a transitive set. Then

$$\begin{aligned} x \in \bigcup A &\Rightarrow \exists b\,(x \in b \wedge b \in A) && \text{(by definition of } \bigcup A\text{)} \\ &\Rightarrow x \in A && \text{(because } A \text{ is transitive)} \end{aligned}$$

ii) (If) Let $\bigcup A \subseteq A$. Then

$$\begin{aligned} x \in a \wedge a \in A &\Rightarrow x \in \bigcup A && \text{(by definition of } \bigcup A\text{)} \\ &\Rightarrow x \in A && \text{(because } \bigcup A \subseteq A\text{)} \end{aligned}$$

■

Transitive Sets

Theorem

A set A is a transitive set iff $a \in A$ implies $a \subseteq A$.

Transitive Sets

Theorem

A set A is a transitive set iff $a \in A$ implies $a \subseteq A$.

Proof.

i) (Only if) Let A be a transitive set and let $a \in A$. If $x \in a$ implies $x \in A$ because A is transitive.
ii) (If) Let $a \in A$ implies $a \subseteq A$. If $x \in a \wedge a \in A$ implies $x \in A$ because $a \subseteq A$.

∎

Transitive Sets

Theorem

A set A is a transitive set iff $A \subseteq \mathcal{P}A$.

Transitive Sets

On transitive sets

Let A be a set. Transitive sets can be defined using any of the followings equivalent affirmations:

(i) $x \in a \in A$ implies $x \in A$,

(ii) $\bigcup A \subseteq A$,

(iii) $a \in A$ implies $a \subseteq A$,

(iv) $A \subseteq \mathcal{P}A$.

Transitive Sets

Theorem 4E

If a is a transitive set, then $\bigcup(a^+) = a$.

Transitive Sets

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Theorem 4F

Every natural number is a transitive set.

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Theorem 4E

If a is a transitive set, then $\bigcup(a^+) = a$.

Theorem 4F

Every natural number is a transitive set.

Theorem 4G

The set ω is a transitive set.

Recursion on Natural Numbers

Recursion theorem on ω (p. 73)

Let A be a set, $a \in A$ and $F : A \to A$. Then there exists a unique function h such that

$$\begin{aligned} h &: \omega \to A \\ h(0) &= a, \\ h(n^+) &= F(h(n)), \text{ for all } n \in \omega. \end{aligned}$$

Arithmetic

Idea

We shall apply the recursion theorem to define addition and multiplication on ω.

Arithmetic

Example

We want to define the function

$$A_5 : w \to w$$
$$A_5(n) = \text{addition of } 5 \text{ to } n.$$

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$$A_5 : w \to w$$
$$A_5(n) = \text{addition of } 5 \text{ to } n.$$

Let $F : \omega \to \omega := n \mapsto n^+$. By the recursion theorem there exists a unique function

$$A_5 : w \to w$$
$$A_5(0) = 5,$$
$$A_5(n^+) = (A_5(n))^+.$$

Arithmetic

Example

Let $m \in \omega$. By the recursion theorem there exists a unique function

$$A_m : w \to w$$
$$A_m(0) = m,$$
$$A_m(n^+) = (A_m(n))^+.$$

Arithmetic

Definition

Let m and n be natural numbers. We define the **addition** of m and n by

$$(+) : w \times w \to w$$
$$m + n = A_m(n).$$

Arithmetic

Theorem 4I

Let m and n be natural numbers. Then

$$\begin{aligned} m + 0 \quad &= m, \\ m + n^+ &= (m + n)^+. \end{aligned}$$

Arithmetic

Example

Let $m \in \omega$. By the recursion theorem there exists a unique function

$$
\begin{aligned}
M_m : w &\to w \\
M_m(0) &= 0, \\
M_m(n^+) &= M_m(n) + m.
\end{aligned}
$$

Arithmetic

Definition

Let m and n be natural numbers. We define the **multiplication** of m and n by

$$\begin{aligned}(\cdot) : w \times w &\to w \\ m \cdot n &= M_m(n).\end{aligned}$$

Arithmetic

Theorem 4J

Let m and n be natural numbers. Then

$$\begin{aligned} m \cdot 0 &= 0, \\ m \cdot n^+ &= (m \cdot n) + m. \end{aligned}$$

Ordering on Natural Numbers

Strong induction principle for ω (p. 87)

Let A be a subset of ω, and assume that for every n in ω,

$$m < n \to m \in A \quad \text{implies} \quad n \in A.$$

Then $A = \omega$.

References

Aczel, Peter (1977). An Introduction to Inductive Definitions. In: Handbook of Mathematical Logic. Ed. by Barwise, Jon. Vol. 90. Studies in Logic and the Foundations of Mathematics. Elsevier. Chap. C.7. DOI: 10.1016/S0049-237X(08)71120-0 (cit. on p. 26).

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Hrbacek, Karel and Jech, Thomas [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on pp. 8, 23, 24).

Mendelson, Elliott [1973] (2008). Number Systems and the Foundations of Analysis. Republication of the 1985 reprint. Dover Publications (cit. on pp. 21, 22).