

CM0832 Elements of Set Theory

6. Cardinal Numbers and the Axiom of Choice

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Enderton 1977].

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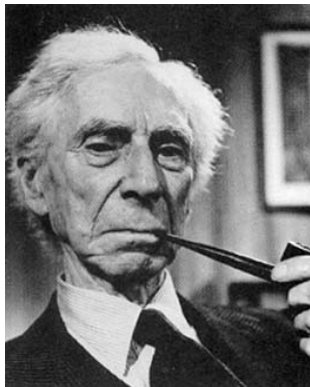
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Example

Whiteboard.



‘The possibility that whole and part may have the same number of terms is, it must be confessed, shocking to common-sense.’ [Russell 1903, p. 358]

Equinumerosity

Theorem 6B(a)

The set ω is not equinumerous to the set $\mathbb{R}$ of real numbers.

Equinumerosity

Proof.

Let's suppose $\omega \approx \mathbb{R}$, that is, there is an one-to-one correspondence $f : \omega \rightarrow \mathbb{R}$ such that*

$$f(0) = 236.001\dots,$$

$$f(1) = -7.777\dots,$$

$$f(2) = 3.1415\dots,$$

$$\vdots$$

(continued on next slide)

*'We assume that a decimal expansion does not contain only the digit 9 from some place on, so each real number has a unique decimal expansion.' [Hrbacek and Jech (1978) 1999, Theorem 6.1, p. 90]

Equinumerosity

Proof.

Let $x = 0.d_1d_2d_3 \dots \in \mathbb{R}$, where

$$d_{n+1} = \begin{cases} 4, & \text{if } f(n) \neq 4; \\ 5, & \text{if } f(n) = 4. \end{cases}$$

The number x does not belong to the above enumeration. Therefore, $\mathbb{R}$ is non-enumerable. ■

On Refutations of Cantor's Diagonal Argument



'I dedicate this essay to the two-dozen-odd people whose refutations of Cantor's diagonal argument have come to me either as referee or as editor in the last twenty years or so... A few years ago it occurred to me to wonder why so many people devote so much energy to refuting this harmless little argument—what had it done to make them angry with it?... These pages report the results.' [Hodges 1998, p. 1]

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Corollary 6D

- (i) Any set equinumerous to a proper subset of itself is infinite.
- (ii) The set ω is infinite.

The Continuum Hypothesis

The continuum hypothesis (CH)

There is no a set whose cardinality is strictly between the cardinality of the set of the natural numbers and the cardinality of the set of real numbers, that is,

$$2^{\aleph_0} = \aleph_1.$$

CH could not be **disproved** [Gödel 1938] nor **proved** [Cohen 1963] in ZFC, that is, CH is independent of ZFC set theory.

References



Cohen, Paul J. (1963). The Independence of the Continuum Hypothesis. Proceedings of the National Academy of Sciences of the United States of America 50.6, pp. 1143–1148. DOI: [10.1073/pnas.50.6.1143](https://doi.org/10.1073/pnas.50.6.1143) (cit. on p. 15).



Enderton, Herbert B. (1977). Elements of Set Theory. Academic Press (cit. on p. 2).



Gödel, Kurt (1938). The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis. Proceedings of the National Academy of Sciences of the United States of America 24.12, pp. 556–557. DOI: [10.1073/pnas.24.12.556](https://doi.org/10.1073/pnas.24.12.556) (cit. on p. 15).



Hodges, Wilfrid (1998). An Editor Recalls some Hopeless Papers. The Bulletin of Symbolic Logic 4.1, pp. 1–16. DOI: [10.2307/421003](https://doi.org/10.2307/421003) (cit. on p. 10).



Hrbacek, Karel and Jech, Thomas [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 8).



Russell, Bertrand (1903). The Principles of Mathematics. Cambridge University Press (cit. on p. 6).