CM0832 Elements of Set Theory
2. Axioms and Operations

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Enderton 1977].

Extensionality Axiom

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If two sets have exactly the same members, then they are equal, that is,

$$\forall A \forall B \,[\forall x \,(x \in A \leftrightarrow x \in B) \rightarrow A = B\,].$$

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$$\forall A \forall B \, [\forall x \, (x \in A \leftrightarrow x \in B) \to A = B \,].$$

Question

Have we any set? No, we haven't.

Some Axioms for Building Sets

Empty set axiom

There is a set having no members, that is,

$$\exists B\, \forall x\, (x \notin B).$$

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Observation

The empty set axiom is equivalent to

$$\exists B \, \forall x \, (x \in B \leftrightarrow x \neq x).$$

Some Axioms for Building Sets

Pairing axiom

For any sets u and v, there is a set having as members just u and v, that is,

$$\forall a \, \forall b \, \exists C \, \forall x \, (x \in C \leftrightarrow x = a \lor x = b).$$

Some Axioms for Building Sets

Union axiom (first version)

For any sets a and b, there is a set whose members are those sets belonging either to a or to b (or both), that is,

$$\forall a \, \forall b \, \exists B \, \forall x \, (x \in B \leftrightarrow x \in a \vee x \in b).$$

Some Axioms for Building Sets

Power set axiom

For any set a, there is a set whose members are exactly the subsets of a, that is,

$$\forall a \, \exists B \, \forall x \, (x \in B \leftrightarrow x \subseteq a),$$

where

$$u \subseteq v := \forall t \, (t \in u \to t \in v).$$

Definitions from Set Abstraction

Observation

Recall that our set of non-logical symbols is $\mathfrak{L} = \{\epsilon\}$. When we add some definitions, we formally are changing this set (e.g. $\mathfrak{L} = \{\epsilon, \emptyset, \cup\}$). See, e.g. [Kunen (2011) 2013, § I.2], [Kunen (1980) 1992, § I.8 and § I.13] and [Suppes (1960) 1972, § 2.1] for how to add valid definitions and how to handle the new sets of non-logical symbols created by these definitions.

Definitions from Set Abstraction

Definitions from the empty, pairing, union and power set axioms via set abstraction

Let a, b, u and v be sets, then we define

$$\begin{aligned}
\emptyset &:= \{\, x \mid x \neq x \,\} && \text{(empty set)},\\
\{u, v\} &:= \{\, x \mid x = u \lor x = v \,\} && \text{(pair set)},\\
\{u\} &:= \{u, u\} && \text{(singleton set)},\\
a \cup b &:= \{\, x \mid x \in a \lor x \in b \,\} && \text{(union)},\\
\mathcal{P}a &:= \{\, x \mid x \subseteq a \,\} && \text{(power set)}.
\end{aligned}$$

Subset Axiom Scheme

Introduction

Whiteboard.

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Subset axiom scheme (axiom scheme of comprehension/separation)

For any propositional function $\varphi(x)$, not containing B, the following is an axiom:

$$\forall c\, \exists B\, \forall x\, (x \in B \leftrightarrow x \in c \wedge \varphi(x)).$$

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We stated an axiom scheme.

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Abstraction from the subset axiom scheme

$\{ \, x \in c \mid \varphi(x) \, \}$ is the set of all $x \in c$ satisfying the property φ.

Subset Axiom Scheme

Observation

The propositional function φ can depend on other variables $t_1, \ldots, t_k$. In this case, we use $\varphi(x, t_1, \ldots, t_k)$ and we universally quantify on variables $t_1, \ldots, t_k$ when using the axiom scheme.

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Theorem 2A

There is no set to which every set belongs.

Proof

Whiteboard.

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Theorem 2A

There is no set to which every set belongs.

Proof

Whiteboard.

Exercise

Why does the subset axiom scheme avoid the Berry paradox?

Arbitrary Unions

Union axiom (final version)

For any set A, there exists a set B whose elements are exactly the members of the members of A, that is,

$$\forall A \, \exists B \, \forall x \, [\, x \in B \leftrightarrow \exists b \, (x \in b \land b \in A) \,].$$

Arbitrary Unions

Definition

Let A be a set. The **union** $\bigcup A$ of A is defined by

$$\bigcup A := \{\, x \mid \exists b \, (x \in b \land b \in A) \,\}.$$

Example (informal)

Let $A = \{\{2, 4, 6\}, \{6, 16, 26\}, \{0\}\}$, then

$$\bigcup A = \{0, 2, 4, 6, 16, 26\}.$$

Example

$$a \cup b = \bigcup \{a, b\}, \qquad \bigcup \{a\} = a, \qquad \bigcup \emptyset = \emptyset.$$

Arbitrary Intersections

Theorem 2B

For any non-empty set A, there exists a unique set B such that for any x,

$$x \in B \quad \text{iff} \quad x \text{ belongs to every member of } A.$$

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Definition

Let A be a non-empty set. The **intersecction** $\bigcap A$ of A can be defined by

$$\bigcap A := \{\, x \mid \forall b\,(b \in A \to x \in b)\,\}, \text{ for } A \neq \emptyset.$$

Algebra of Sets

Exercise 2.18

Assume that A and B are subsets of S. List all of the different sets that can be made from these three by use of the binary operations $\cup$, $\cap$, and $-$.

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The Venn diagram shows four possible regions for shading, that is, we have 2^4 different sets given by

$\emptyset$, A, B, S, $A \cup B$, $A \cap B$, $A - B$, $B - A$, $A + B$, $S - A$, $S - B$, $S - (A \cup B)$, $S - (A \cap B)$, $S - (A - B)$, $S - (B - A)$ and $S - (A + B)$,

where the binary operation $+$ is the **symmetric difference** defined by

$$\begin{aligned} A + B &:= (A - B) \cup (B - A) \\ &= (A \cup B) - (A \cap B). \end{aligned}$$

References

- Enderton, Herbert B. (1977). Elements of Set Theory. Academic Press (cit. on p. 2).
- Kunen, Kenneth [1980] (1992). Set Theory. An Introduction to Independence Proofs. 5th impression. North-Holland (cit. on p. 10).
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- Suppes, Patrick [1960] (1972). Axiomatic Set Theory. Corrected republication. Dover Publications (cit. on p. 10).