

CM0832 - MT5001 Elements of Set Theory

Introduction to the ZFC Theory

Andrés Sicard-Ramírez

Universidad EAFIT

Semester 2026-2

Preliminaries

Textbook

Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Preliminaries

Notation

$\mathbb{N}$: Set of natural numbers (including the zero)

$\mathbb{Z}$: Set of all integers (positive, negative and zero)

$\mathbb{R}$: Set of real numbers

Preliminaries

Notation

$\mathbb{N}$: Set of natural numbers (including the zero)

$\mathbb{Z}$: Set of all integers (positive, negative and zero)

$\mathbb{R}$: Set of real numbers

Acronyms

ZFC (Zermelo–Fraenkel set theory with Choice)

FOLEQ (First-Order Logic with Equality)

Outline

Defining Sets by Extension and by Intension

Russell's Paradox

The Axiom of Choice

References

Outline

Defining Sets by Extension and by Intension

Russell's Paradox

The Axiom of Choice

References

Defining Sets by Extension and by Intension

Description

In naive set theory we can define sets by **extension** (explicitly listing its elements) or by **intension** (given a **property** of its elements).

Defining Sets by Extension and by Intension

Example (finite set)

$\{ 2, 4, 6, 8, 10, 12, 14 \},$

(by extension);

$\{ x \in \mathbb{N} : 0 < x < 15 \text{ and } x \text{ is even} \},$

(by intension).

Defining Sets by Extension and by Intension

Example (finite set)

$\{ 2, 4, 6, 8, 10, 12, 14 \},$ (by extension);

$\{ x \in \mathbb{N} : 0 < x < 15 \text{ and } x \text{ is even} \},$ (by intension).

Example (infinite set)

The infinite sets are defined by intension via a **finite description**.

$\{ \dots, -4, -2, 0, 2, 4, \dots \},$ (by extension);

$\{ x \in \mathbb{Z} : x \text{ is even} \},$ (by intension).

Defining Sets by Extension and by Intension

Example

What about sets without an “elegant” finite description? No problem!

$\{ 1, 7, 42 \},$ (by extension);

$\{ x : x = 1 \text{ or } x = 7 \text{ or } x = 42 \},$ (by intension).

Defining Sets by Extension and by Intension

A key feature of the set theory and mathematics [...] is that every mathematical object that we discuss can indeed be described in a finite way. As the most exciting part of set theory deals with infinite sets, and as such a set can never be finitely described by extension, definition by intension is crucial. (p. 66)

Defining Sets by Extension and by Intension

Description

Let P be a **property**. In general, we can define a set S by intension by

$$S = \{ x : x \text{ has property } P \},$$

Defining Sets by Extension and by Intension

Description

Let P be a **property**. In general, we can define a set S by intension by

$$S = \{ x : x \text{ has property } P \},$$

Remark

In the literature the above description is called **the unrestricted (or naive) comprehension principle**.

Defining Sets by Extension and by Intension

Description

Let P be a **property**. In general, we can define a set S by intension by

$$S = \{ x : x \text{ has property } P \},$$

Remark

In the literature the above description is called **the unrestricted (or naive) comprehension principle**.[†]

[†]See, i.e., (Button 2021; Enderton 1977).

Defining Sets by Extension and by Intension

Example

Defining a set by intension allows some **circularities**.

Let $\llbracket 1 \rrbracket$ be the set defined by

$$\llbracket 1 \rrbracket = \{ x : x \text{ has one element} \},$$

and be the singleton set $\{ \llbracket 1 \rrbracket \}$. Then

$$\llbracket 1 \rrbracket \in \{ \llbracket 1 \rrbracket \} \in \llbracket 1 \rrbracket.$$

Defining Sets by Extension and by Intension

Example

Defining a set by intension allows some **circularities**.

Let $\llbracket 1 \rrbracket$ be the set defined by

$$\llbracket 1 \rrbracket = \{ x : x \text{ has one element} \},$$

and be the singleton set $\{ \llbracket 1 \rrbracket \}$. Then

$$\llbracket 1 \rrbracket \in \{ \llbracket 1 \rrbracket \} \in \llbracket 1 \rrbracket.$$

Question

Is there any problem with the above circularity?

Outline

Defining Sets by Extension and by Intension

Russell's Paradox

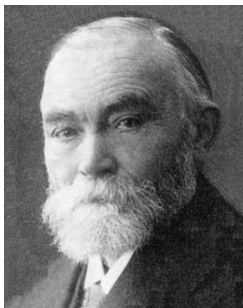
The Axiom of Choice

References

Russell's Paradox

Historical remark

The paradox was communicated in 1901 by the British mathematician and philosopher Bertrand Russell to the German mathematician and philosopher Gottlob Frege, undermining Frege's attempt to reduce mathematics to logic (images from Wikipedia)



Gottlob Frege (1848 – 1925)



Bertrand Russell (1872 – 1970)

Russell's Paradox

Description

Let $\mathcal{R}$ be the set defined by

$$\mathcal{R} = \{ x : x \notin x \}.$$

Russell's Paradox

Description

Let $\mathcal{R}$ be the set defined by

$$\mathcal{R} = \{ x : x \notin x \}.$$

Does $\mathcal{R} \in \mathcal{R}$?

Russell's Paradox

Description

Let $\mathcal{R}$ be the set defined by

$$\mathcal{R} = \{ x : x \notin x \}.$$

Does $\mathcal{R} \in \mathcal{R}$?

- If $\mathcal{R} \in \mathcal{R}$ then $\mathcal{R} \notin \mathcal{R}$.
- If $\mathcal{R} \notin \mathcal{R}$ then $\mathcal{R} \in \mathcal{R}$.

Russell's Paradox

Consequences

Recall from the previous lecture that Russell's paradox (and other paradoxes) generated a **crisis** in the foundations of the mathematics.

Russell's Paradox

Possible solutions

- Hierarchy of types (Russell and Whitehead)
- Restrict the ways in which construct new sets via the ZFC theory (Zermelo, Fraenkel, Skolem and other).

Carta of Russell a van Heijenoort

Penrhyndeudraeth, 23 de noviembre de 1962

Estimado Profesor van Heijenoort,

Cuando pienso en actos de gracia e integridad, me doy cuenta de que no conozco ninguno comparable con la dedicación de Frege a la verdad. Estaba Frege dando cima a la obra de toda su vida, la mayor parte de su trabajo había sido ignorado en beneficio de hombres infinitamente menos competentes que él, su segundo volumen estaba a punto de ser publicado y, al darse cuenta de que su supuesto fundamental era erróneo, reaccionó con placer intelectual, reprimiendo todo sentimiento de decepción personal. Era algo casi sobrehumano y un índice de aquello de lo que los hombres son capaces cuando están dedicados al trabajo creador y al conocimiento, y no al crudo afán por dominar y hacerse famosos.

Cordialmente,

Bertrand Russell

Carta of Russell a van Heijenoort[†]

Penrhyndeudraeth, 23 de noviembre de 1962

Estimado Profesor van Heijenoort,

Cuando pienso en actos de gracia e integridad, me doy cuenta de que no conozco ninguno comparable con la dedicación de Frege a la verdad. Estaba Frege dando cima a la obra de toda su vida, la mayor parte de su trabajo había sido ignorado en beneficio de hombres infinitamente menos competentes que él, su segundo volumen estaba a punto de ser publicado y, al darse cuenta de que su supuesto fundamental era erróneo, reaccionó con placer intelectual, reprimiendo todo sentimiento de decepción personal. Era algo casi sobrehumano y un índice de aquello de lo que los hombres son capaces cuando están dedicados al trabajo creador y al conocimiento, y no al crudo afán por dominar y hacerse famosos.

Cordialmente,

Bertrand Russell

[†]La carta fue escrita en 1962. La versión original en inglés está en (van Heijenoort 1967, pág. 127). La versión en español está en (Frege [1971] 1973, págs. 8–9). La carta fue traducida por Jesús Mosterín.

Outline

Defining Sets by Extension and by Intension

Russell's Paradox

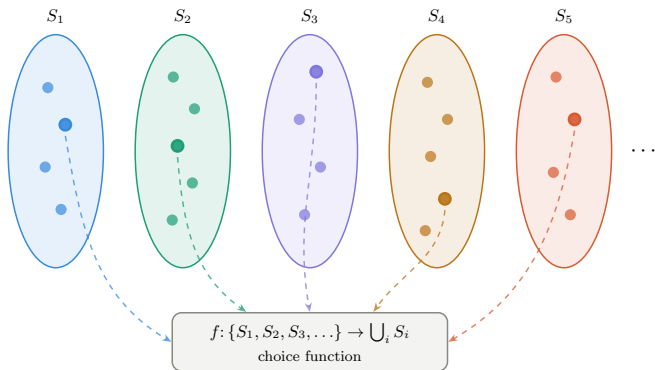
The Axiom of Choice

References

The Axiom of Choice

Description

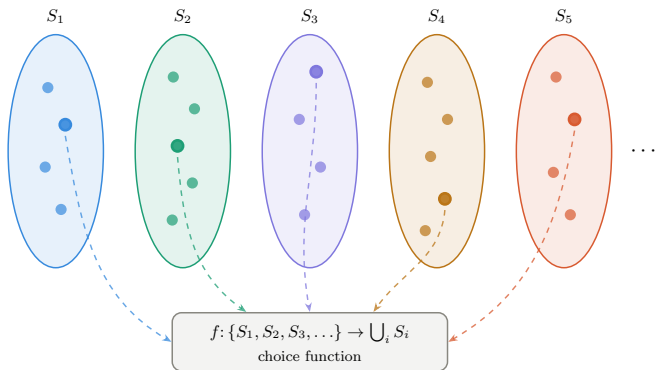
From any collection (finite or infinite) of non-empty sets, there exists a choice function that picks one element from each set.



The Axiom of Choice

Description

From any collection (finite or infinite) of non-empty sets, there exists a choice function that picks one element from each set.[†]



[†] $\text{\LaTeX}$ figure generated by Claude Code.

The Axiom of Choice

Remark

A very complete history of origins, development and influence of the axiom of choice is in (Moore 1982).

The Axiom of Choice

About the ZFC theory and the axiom of choice

“The axioms are based on those proposed by the German mathematician Ernst Zermelo (1871-1953) in 1905. It is important to realize, however, that Zermelo’s motivation for producing axioms was not so much the avoidance of paradoxes, but primarily to give a framework for a controversial result about sets, of much interest to mathematicians of the day, and still of great significance. This result concerned a problem of Cantor’s, whether it was possible to define a well-ordering on the set of real numbers, and Zermelo’s argument, exploiting a principle called the axiom of choice, had previously been rejected by other mathematicians. By enunciating the assumptions about sets underlying his argument, Zermelo hoped to make it harder for others to reject it.” (pp. 69–70)

Outline

Defining Sets by Extension and by Intension

Russell's Paradox

The Axiom of Choice

References

References

- Tim Button (2021). Set Theory. An Open Introduction. Fall 2021. Revision 2138a50 (2026-07-11). Independently published. URL: <https://builds.openlogicproject.org/courses/set-theory/> (cit. on p. 14).
- Herbert B. Enderton (1977). Elements of Set Theory. Academic Press (cit. on p. 14).
- Gottlob Frege [1971] (1973). Estudios sobre Semántica. Trans. by Ulises Moulines. 2nd ed. Vol. 60. Ariel quincenal. Editorial Ariel, S.A. (cit. on p. 25).
- Derek Goldrei (1996). Classic Set Theory. A Guided Independent Study. Chapman & Hall (cit. on p. 2).
- Gregory H. Moore (1982). Zermelo's Axiom of Choice. Its Origins, Development, and Influence. Vol. 8. Studies in the History of Mathematics and Physical Sciences. Springer-Verlag. DOI: [10.1007/978-1-46](https://doi.org/10.1007/978-1-46) (cit. on p. 29).
- Jean van Heijenoort, ed. (1967). From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Source Books in the History of the Sciences. Harvard University Press (cit. on p. 25).