

CM0832 - MT5001 Elements of Set Theory

The Recursion Theorem

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

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Introduction

“Our next task is to show how to define addition, multiplication, and other familiar operations of arithmetic. To facilitate this, we develop an important general method for defining functions on $\mathbf{N}$.” (p. 46)

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Sequences

Definition (p. 46)

A **sequence** is a function whose domain is either

- (a) a natural number n or
- (b) the set $\mathbf{N}$.

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Definitions (p. 46)

- (a) A **finite sequence of length** n is a sequence whose domain is $n \in \mathbb{N}$.
- (b) An **infinite sequence** is a sequence whose domain is $\mathbb{N}$.

Sequences

Notation (p. 46)

- A finite sequence of length n is denoted by

$$\langle a_i \mid i < n \rangle \quad \text{or} \quad \langle a_i \mid i \doteq 0, 1, \dots, n-1 \rangle \quad \text{or} \quad \langle a_0, a_1, \dots, a_{n-1} \rangle.$$

- The range of a finite sequence of length n is denoted by

$$\{ a_i \mid i \in \mathbf{N} \} \quad \text{or} \quad \{ a_0, a_1, \dots, a_{n-1} \}.$$

Sequences

Notation (p. 46)

- An infinite sequence is denoted by

$$\langle a_i \mid i \in \mathbf{N} \rangle \quad \text{or} \quad \langle a_i \mid i = 0, 1, \dots \rangle \quad \text{or} \quad \langle a_i \rangle_{i=0}^{\infty} .$$

- The range of an infinite sequence is denoted by

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Sequences

Example

The **empty sequence** $\langle \rangle$ is the finite sequence of length 0.

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Recall that B^A denotes the set of all functions on A into B .

The set of all finite sequences of elements of A , denoted $\text{Seq}(A)$, is defined by

$$\text{Seq}(A) \stackrel{\text{def}}{=} \bigcup_{n \in \mathbf{N}} A^n.$$

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Example

Note that an infinite sequence of elements of A is an element of $A^{\mathbb{N}}$.

Sequences

Example

- A sequence $s : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\begin{aligned}s_0 &= 1, \\ s_{n+1} &= n^2, \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Sequences

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- From the definition of s we could define a property $\mathbf{P}(x, y)$ such that

$$s_x \doteq y \quad \text{if and only if} \quad \mathbf{P}(x, y).$$

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- For proving that s is a sequence (existence and uniqueness) we could define a property $\mathbf{P}(x, y)$: “either $x \doteq 0$ and $y \doteq 1$ or, for some $n \in \mathbf{N}$, $x + 1$ and $y \doteq n^2$ ”, and use the axioms on

$$s = \{ (x, y) \in \mathbf{N} \times \mathbf{N} \mid \mathbf{P}(x, y) \}.$$

Sequences

Example

A sequence $f : \mathbf{N} \rightarrow \mathbf{N}$ is defined by

$$\begin{aligned} f_0 &= 1, \\ f_{n+1} &= f_n \times (n + 1), \text{ for all } n \in \mathbf{N}. \end{aligned}$$

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A sequence $f : \mathbf{N} \rightarrow \mathbf{N}$ is defined by

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Question

From the definition of f how we could define a property $\mathbf{P}(x, y)$ such that

$$f_x \doteq y \quad \text{if and only if} \quad \mathbf{P}(x, y)$$

?

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Recursion on Natural Numbers

The Recursion Theorem (p. 47)

For any set A , any $a \in A$, and any function $g : A \times \mathbf{N} \rightarrow A$, there exists a unique infinite sequence $f : \mathbf{N} \rightarrow A$ such that

- (a) $f_0 = a$,
- (b) $f_{n+1} = g(f_n, n)$, for all $n \in \mathbf{N}$.

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- (a) $f_0 = a$,
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Remark

The set a is the “initial value” of f and the function g provides instructions for computing the value f_{n+i} assuming that the value f_n has been computed.

Recursion on Natural Numbers

Example

We want to define the function

$$A^5 : \mathbf{N} \rightarrow \mathbf{N}$$
$$A^5(n) = \text{addition of } 5 \text{ to } n.$$

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Let $A = \mathbf{N}$, $a = 5$ and the function $g : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ defined by $g(u, v) = u + 1$. By the Recursion Theorem there exists a unique sequence $f : \mathbf{N} \rightarrow \mathbf{N}$

$$f_0 = 5,$$
$$f_{n+1} = f_n + 1, \text{ for all } n \in \mathbf{N}.$$

where $A^5 = f$.

Recursion on Natural Numbers

Example

For each $m \in \mathbf{N}$ by the Recursion Theorem there exists a unique function

$$A^m : \mathbf{N} \rightarrow \mathbf{N}$$
$$A^m(n) = \text{addition of } m \text{ to } n.$$

Recursion on Natural Numbers

Example

Let m and n be natural numbers. We could define the **addition** of m and n , denoted $\text{add}(m, n)$, by

$$\text{add} : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$$

$$\text{add}(m, n) = A^m(n).$$

Recursion on Natural Numbers

Theorem [3.]3.6 (Parametric version of the Recursion Theorem)

Let P and A be sets and let $a : P \rightarrow A$ and $g : P \times A \times \mathbf{N} \rightarrow A$ be functions. There exists a unique function $f : P \times \mathbf{N} \rightarrow A$ such that

- (a) $f(p, 0) = a(p)$, for all $p \in P$.
- (b) $f(p, n + 1) = g(p, f(p, n), n)$, for all $p \in P$ and $n \in \mathbf{N}$.

Recursion on Natural Numbers

Example

By the parametric version of the Recursion Theorem, with $P = A = \mathbf{N}$, $a(p) = p$, for all $p \in \mathbf{N}$ and $g(p, x, n) = x + 1$, for all $p, x, n \in \mathbf{N}$; there is a unique function $+: \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ such that

- (a) $+(m, 0) = m$, for all $m \in \mathbf{N}$.
- (b) $+(m, n + 1) = +(m, n) + 1$, for all $m, n \in \mathbf{N}$.

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