CM0832 - MT5001 Elements of Set Theory
The Formal Language of Zermelo–Fraenkel Set Theory with Choice

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Preliminaries

Acronyms

ZFC	(Zermelo–Fraenkel set theory with Choice)
FOLEQ	(First-Order Logic with Equality)

Outline

First-Order Logic

The ZFC Theory

Properties

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First-Order Logic

Theorem

The identity (equality) relation is an equivalence relation.

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The identity (equality) relation is an equivalence relation.
That is, for all X, Y, Z,

(a) $X \doteq X$ (reflexivity),

(b) if $X \doteq Y$, then $Y \doteq X$ (symmetry),

(c) if $X \doteq Y$ and $Y \doteq Z$, then $X \doteq Z$ (transitivity).

First-Order Logic

Definition

A **proposition** (or **statement**) is a sentence that can be assigned a truth value of true or false.

First-Order Logic

Definitions

A **propositional function** (or **property**) is a sentence containing one or more variables (parameters) that becomes a proposition when specific values are assigned to its variables.

First-Order Logic

Definitions

A **propositional function** (or **property**) is a sentence containing one or more variables (parameters) that becomes a proposition when specific values are assigned to its variables.

The **domain** of a propositional function is the domain of the discourse.

The **codomain** of a propositional function is $\{\text{true}, \text{false}\}$.

First-Order Logic

Theorem

The identity (equality) relation is substitutive.

First-Order Logic

Theorem

The identity (equality) relation is substitutive.
That is, let $\mathbf{P}(X)$ be a propositional function. For all X, Y, if $X \doteq Y$ and $\mathbf{P}(X)$, then $\mathbf{P}(Y)$.

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Pure Sets

Definition

A **pure set** is a set such that its members are also sets.

Pure Sets

Definition

A **pure set** is a set such that its members are also sets.

Description

The ZFC theory is a first-order theory of pure sets.

First-Order Language of ZFC

Description

In our informal but axiomatic presentation of ZFC we have two primitive (undefined) concepts: set and membership relation.

First-Order Language of ZFC

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In our informal but axiomatic presentation of ZFC we have two primitive (undefined) concepts: set and membership relation.

The (binary) membership relation is denoted $\in$.

First-Order Language of ZFC

Notation

$$X \dot{\neq} Y \stackrel{\text{def}}{=} \neg(X \doteq Y),$$

$$X \notin Y \stackrel{\text{def}}{=} \neg(X \in Y).$$

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Properties

Example

We define the following unary property:

$\mathbf{P}(X)$: "There exists no $Y \in X$".

Properties

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We define the following unary property:

$\mathbf{P}(X)$: "There exists no $Y \in X$".

Since that we can prove that there is a unique set X with the property $\mathbf{P}(X)$ we can add a new constant symbol $\emptyset$ denoting the empty set.

Properties

Example

We define the following binary property:

$\mathbf{P}(X, Y)$: "Every element of X is an element of Y".

Properties

Example

We define the following binary property:

$\mathbf{P}(X, Y)$: "Every element of X is an element of Y".

Using the above property we can add a new binary relation symbol $\subseteq$ denoting that X is a subset of Y:

$$X \subseteq Y \stackrel{\text{def}}{=} \mathbf{P}(X, Y).$$

Properties

Example

We define the following ternary property:

$\mathbf{Q}(X, Y, Z)$: "For every U, $U \in Z$ if and only if $U \in X$ and $U \in Y$".

Properties

Example

We define the following ternary property:

$\mathbf{Q}(X, Y, Z)$: "For every U, $U \in Z$ if and only if $U \in X$ and $U \in Y$".

Since that we can prove that for every X and Y there is a unique Z such that $\mathbf{Q}(X, Y, Z)$, we can add a new binary function symbol $\cap$, denoting the intersection of X and Y:

$$X \cap Y \stackrel{\text{def}}{=} \text{the unique } Z \text{ such that } \mathbf{Q}(X, Y, Z).$$

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References

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 2).