CM0832 - MT5001 Elements of Set Theory
1. Sets

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Acronyms

ZFC Zermelo-Fraenkel set theory with Choice
FOL First-Order Logic

First-Order Logic

Theorem

The identity (equality) relation is an equivalence relation.

First-Order Logic

Theorem

The identity (equality) relation is an equivalence relation.
That is, for all X, Y, Z,

(i) $X \doteq X$ (reflexivity),

(ii) if $X \doteq Y$, then $Y \doteq X$ (symmetry),

(iii) if $X \doteq Y$ and $Y \doteq Z$, then $X \doteq Z$ (transitivity).

First-Order Logic

Definition

A **proposition** (or **statement**) is a sentence that can be assigned a truth value of true or false.

First-Order Logic

Definitions

A **propositional function** (or **property**) is a sentence containing one or more variables (parameters) that becomes a proposition when specific values are assigned to its variables.

First-Order Logic

Definitions

A **propositional function** (or **property**) is a sentence containing one or more variables (parameters) that becomes a proposition when specific values are assigned to its variables.

The **domain** of a propositional function is the domain of the discourse.

The **codomain** of a propositional function is $\{\text{true}, \text{false}\}$.

First-Order Logic

Theorem

The identity (equality) relation is substitutive.

First-Order Logic

Theorem

The identity (equality) relation is substitutive.
That is, let $\mathbf{P}(X)$ be a propositional function. For all X, Y, if $X \doteq Y$ and $\mathbf{P}(X)$, then $\mathbf{P}(Y)$.

First-Order Language of ZFC

Description

In our informal but axiomatic presentation of ZFC we have two primitive (undefined) concepts: set and membership relation.

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In our informal but axiomatic presentation of ZFC we have two primitive (undefined) concepts: set and membership relation.

The (binary) membership relation is denoted by $\in$.

First-Order Language of ZFC

Notation

$$X \mathrel{\dot{\neq}} Y \stackrel{\text{def}}{=} \neg(X \doteq Y),$$

$$X \notin Y \stackrel{\text{def}}{=} \neg(X \in Y).$$

Properties

Example

We define the following unary property:

$\mathbf{P}(X)$: "There exists no $Y \in X$".

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We define the following unary property:

$\mathbf{P}(X)$: "There exists no $Y \in X$".

Since that we can prove that there is a unique set X with the property $\mathbf{P}(X)$ we can add a new constant symbol $\emptyset$ denoting the empty set.

Properties

Example

We define the following binary property:

$\mathbf{P}(X, Y)$: "Every element of X is an element of Y".

Properties

Example

We define the following binary property:

$\mathbf{P}(X, Y)$: "Every element of X is an element of Y".

Using the above property we can add a new binary relation symbol $\subseteq$ denoting that X is a subset of Y:

$$X \subseteq Y \stackrel{\text{def}}{=} \mathbf{P}(X, Y).$$

Properties

Example

We define the following ternary property:

$\mathbf{Q}(X, Y, Z)$: "For every U, $U \in Z$, if and only if, $U \in X$ and $U \in Y$".

Properties

Example

We define the following ternary property:

$\mathbf{Q}(X, Y, Z)$: "For every U, $U \in Z$, if and only if, $U \in X$ and $U \in Y$".

Since that we can prove that for every X and Y there is a unique Z such that $\mathbf{Q}(X, Y, Z)$, we can add a new binary function symbol $\cap$, denoting the intersection of X and Y:

$$X \cap Y \stackrel{\text{def}}{=} \text{the unique } Z \text{ such that } \mathbf{Q}(X, Y, Z).$$

The Axioms

The Axiom of Existence

- *"There exists a set which has no elements." (Textbook)*
- $(\exists B)(\forall x)(x \notin B).$

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Remark

The Axiom of Existence postulate the our universe of discourse is not void. This axiom postulates the existence of the empty set.

The Axioms

The Axiom of Existence

- *"There exists a set which has no elements." (Textbook)*
- $(\exists B)(\forall x)(x \notin B).$

Remark

The Axiom of Existence postulate the our universe of discourse is not void. This axiom postulates the existence of the empty set.†

†Strictly speaking, the universe of the discourse of FOL is not empty. For example, let $\mathbf{P}(x)$ be a property, then $(\forall x)\mathbf{P}(x) \rightarrow (\exists x)\mathbf{P}(x)$ is a theorem. See, e.g., (Lambert 2001).

The Axioms

The Axiom of Extensionality

- *"If every element of* X *is an element of* Y *and every element of* Y *is an element of* X*, then* $X \doteq Y$*." (Textbook)*
- *"If two sets have exactly the same members, then they are equal." (Enderton 1977)*
- $(\forall X)(\forall Y)[(\forall z)(z \in X \leftrightarrow z \in Y) \to X \doteq Y].$

The Axioms

Lemma (1.)3.1

There exists only one set with no elements.

The Axioms

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There exists only one set with no elements.

Proof

In the lecture.

The Axioms

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There exists only one set with no elements.

Proof

In the lecture.

Definition (1.)3.2

The **empty set**, denoted $\emptyset$, is set with no elements.

The Axioms

Theorem (converse of the Axiom of Extensionality)

If $X \doteq Y$, then every element of X is an element of Y and every element of Y is an element of X.

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If $X \doteq Y$, then every element of X is an element of Y and every element of Y is an element of X.

$$(\forall X)(\forall Y)[\, X \doteq Y \to (\forall z)(z \in X \leftrightarrow z \in Y)\,].$$

Proof

FOL.

The Axioms

Theorem (converse of the Axiom of Extensionality)

If $X \doteq Y$, then every element of X is an element of Y and every element of Y is an element of X.[†]

$$(\forall X)(\forall Y)[\, X \doteq Y \to (\forall z)(z \in X \leftrightarrow z \in Y)\,].$$

Proof

FOL.

[†] Due to this theorem some authors (see, e.g. (Goldrei 1996, p. 76)) state the Axiom of Extensionality using a bi-conditional, that is, if every element of X is an element of Y and every element of Y is an element of X, if and only if, $X \doteq Y$. $(\forall X)(\forall Y)[\,(\forall z)(z \in X \leftrightarrow z \in Y) \leftrightarrow X \doteq Y\,].$

The Axioms

Theorem

The identity (equality) relation is compatible with the membership relation.

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The identity (equality) relation is compatible with the membership relation.
That is, for all X, Y, Z,

(i) if $X \doteq Y$ and $X \in Z$, then $Y \in Z$,

(ii) if $X \doteq Y$ and $Z \in X$, then $Z \in Y$.

The Axioms

Theorem

The identity (equality) relation is compatible with the membership relation.
That is, for all X, Y, Z,

(i) if $X \doteq Y$ and $X \in Z$, then $Y \in Z$,

(ii) if $X \doteq Y$ and $Z \in X$, then $Z \in Y$.

Proof

(i) Hint. To use that the identity is substitutive and the property $\mathbf{P}(A, B)$: "$A \in B$".

(ii) Hint. To use the converse of the Axiom of Extensionality.

The Axioms

The Axiom Schema of Comprehension

- *"Let* $\mathbf{P}(x)$ *be a unary property of* x*. For any set* A*, there is a set* B *such that* $x \in B$*, if and only if,* $x \in A$ *and* $\mathbf{P}(x)$*." (Textbook)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

The Axioms

The Axiom Schema of Comprehension

- *"Let $\mathbf{P}(x)$ be a unary property of x. For any set A, there is a set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$." (Textbook)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Remark

We postulate an axiom schema, i.e., there is an axiom for each property $\mathbf{P}(x)$.

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The Axiom Schema of Comprehension

- *"Let $\mathbf{P}(x)$ be a unary property of x. For any set A, there is a set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$." (Textbook)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Example

Let $\mathbf{P}(x)$: "$x \notin x$". The axiom postulates:

For any set A, there is a set B such that $x \in B$, if and only if, $x \in A$ and $x \notin x$ (the set B is the empty set).

The Axioms

The Axiom Schema of Comprehension

- *"Let* $\mathbf{P}(x)$ *be a unary property of* x. *For any set* A, *there is a set* B *such that* $x \in B$, *if and only if,* $x \in A$ *and* $\mathbf{P}(x)$.*" (Textbook)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Remark

"Although the supply of axioms is unlimited, this causes no problems, since it is easy to recognize whether a particular statement is or is not an axiom and since every proof uses only finitely many axioms." (p. 8)

The Axioms

The Axiom Schema of Comprehension

- *"Let $\mathbf{P}(x)$ be a unary property of x. For any set A, there is a set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$." (Textbook)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Remark

When using the axiom scheme with a $(n+1)$-ary property $\mathbf{P}(x, p_1, \ldots, p_n)$ the axiom is:

For any sets $p_1, \ldots, p_n, A$, there is a set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x, p_1, \ldots, p_n)$.

$$(\forall p_1) \cdots (\forall p_n)(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x, p_1, \ldots, p_n)).$$

The Axioms

Example (1).3.3

To prove that if A and B are sets, then there is a set C such that $x \in C$, if and only if, $x \in A$ and $x \in B$.

The Axioms

Example (1).3.3

To prove that if A and B are sets, then there is a set C such that $x \in C$, if and only if, $x \in A$ and $x \in B$.

Proof

Hint: To define the binary property $\mathbf{P}(x, B)$: "$x \in B$" and use the Axiom Schema of Comprehension.

The Axioms

Question

Let $\mathbf{P}(x)$: "$x \notin x$". Without using the Axiom of Existence can we prove the existence of the empty set from the Axiom Schema of Comprehension and the property $\mathbf{P}(x)$?

The Axioms

Lemma (1.)3.4

For every A, there is only one set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$.

The Axioms

Lemma (1.)3.4

For every A, there is only one set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$.

Remark

This lemma establishes that the set postulated by the Axiom Schema of Comprehension is unique.

The Axioms

Lemma (1.)3.4

For every A, there is only one set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$.

Proof

In the lecture.

The Axioms

Lemma (1.)3.4

For every A, there is only one set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$.

Question

Given the above lemma, why not state the Axiom Schema of Comprehension by "for any set A, there is a unique set B such that $x \in B$, if and only if, $x \in A$ and $\mathbf{P}(x)$"?

$$(\forall A)(\exists! B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x)).$$

The Axioms

Definition (1).3.5

Given a set A and a property $\mathbf{P}(x)$, the set B postulated by the Axiom Schema of Comprehension is

$$\{\, x \in A \mid \mathbf{P}(x) \,\}.$$

The Axioms

Definition (1).3.5

Given a set A and a property $\mathbf{P}(x)$, the set B postulated by the Axiom Schema of Comprehension is

$$\{\, x \in A \mid \mathbf{P}(x) \,\}.^{\dagger}$$

[†] Strictly speaking before this definition we should introduce the abstraction symbol $\{\,\cdot \mid \cdot\,\}$ in the language of ZFC. See, e.g. (Drake 1974).

The Axioms

The Axiom of Pair

- *"For any A and B, there is a set C such that $x \in C$, if and only if, $x \doteq A$ or $x \doteq B$." (Textbook)*
- *"For any sets A and B, there is a set having as members just A and B." (Enderton 1977)*
- $(\forall A)(\forall B)(\exists C)(\forall x)(x \in C \leftrightarrow x \doteq A \vee x \doteq B)$.

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- $(\forall A)(\forall B)(\exists C)(\forall x)(x \in C \leftrightarrow x \doteq A \vee x \doteq B)$.

Exercise

To prove that the set postulated by the axiom is unique.

The Axioms

Definition

The **unordered pair** of A and B, denoted by $\{A, B\}$, is the set postulated by the Axiom of Pair.

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Definition

The **unordered pair** of A and B, denoted by $\{A, B\}$, is the set postulated by the Axiom of Pair.

Notation

If $A \doteq B$ then $\{A\} \overset{\text{def}}{=} \{A, A\}$.

The Axioms

The Axiom of Union

- *"For any set S, there exists a set U such that $x \in U$, if and only if, $x \in A$ for some $A \in S$." (Textbook)*
- *"For any set S, there exists a set U whose elements are exactly the members of the members of A". (Enderton 1977)*
- $(\forall S)(\exists U)(\forall x)[\, x \in U \leftrightarrow (\exists A)(x \in A \wedge A \in S)\,].$

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The Axiom of Union

- *"For any set S, there exists a set U such that $x \in U$, if and only if, $x \in A$ for some $A \in S$." (Textbook)*
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To prove that the set postulated by the axiom is unique.

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The Axiom of Union

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- *"For any set S, there exists a set U whose elements are exactly the members of the members of A". (Enderton 1977)*
- $(\forall S)(\exists U)(\forall x)[\, x \in U \leftrightarrow (\exists A)(x \in A \wedge A \in S) \,]$.

Definition

The **union** of S, denoted by $\bigcup S$, is the set postulated by the axiom.

The Axioms

Example

Let A, B sets. Then, $x \in \bigcup\{A, B\}$, if and only if, $x \in A$ or $x \in B$.

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Example

Let A, B sets. Then, $x \in \bigcup\{A, B\}$, if and only if, $x \in A$ or $x \in B$.

Definition

The **union** of A and B, denoted by $A \cup B$ is the set $\bigcup\{A, B\}$.

The Axioms

Definition (1.)3.10

A is a **subset** of B, denoted by $A \subseteq B$, if and only, if every element of A belongs to B.

The Axioms

The Axiom of Power Set

- *"For any set S, there exists a set P such that $X \in P$, if and only if, $X \subseteq S$."* *(Textbook)*
- *"For any set S, there is a set whose members are exactly the subsets of S."* *(Enderton 1977)*
- $(\forall S)(\exists P)(\forall X)[\,X \in P \leftrightarrow X \subseteq S\,].$

The Axioms

The Axiom of Power Set

- *"For any set S, there exists a set P such that $X \in P$, if and only if, $X \subseteq S$."* *(Textbook)*
- *"For any set S, there is a set whose members are exactly the subsets of S."* *(Enderton 1977)*
- $(\forall S)(\exists P)(\forall X)[X \in P \leftrightarrow X \subseteq S]$.

Exercise

To prove that the set postulated by the axiom is unique.

The Axioms

The Axiom of Power Set

- *"For any set S, there exists a set P such that $X \in P$, if and only if, $X \subseteq S$." (Textbook)*
- *"For any set S, there is a set whose members are exactly the subsets of S." (Enderton 1977)*
- $(\forall S)(\exists P)(\forall X)[X \in P \leftrightarrow X \subseteq S]$.

Definition

The **power set** of S, denoted by $\mathcal{P}(S)$, is the set postulated by the axiom, i.e., is the set of all subsets of S.

Elementary Operations on Sets

Reading

To read Section (1).4. Elementary Operations on Sets.

References

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