CM0832 - MT5001 Elements of Set Theory
Relations

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

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Binary Relations

Definition [2.]2.1

A set R is a **binary relation** if all elements of R are ordered pairs.

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Remark

> *"A binary relation is, therefore, determined by specifying all ordered pairs of objects in that relation; it does not matter by what property the set of these ordered pairs is described." (p. 19)*

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Notation

Let R be a binary relation.

(a) We write $(a, b) \in R$ or aRb.

(b) We write $(a, b) \notin R$ or $a \not R b$.

Binary Relations

Example (informal)

Whiteboard.

Binary Relations

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Example (informal)

Let $\mathbf{N} = \{0, 1, 2, \dots\}$. The identity relation on $\mathbf{N}$, denoted $\mathrm{Id}_{\mathbf{N}}$, is defined by

$$\begin{aligned}\mathrm{Id}_{\mathbf{N}} &= \{\, (n, n) \mid n \in \mathbf{N} \,\} \\ &= \{(0,0), (1,1), (2,2), \dots\}.\end{aligned}$$

Binary Relations

Definition [2.]2.3

Let R be a binary relation. Then

$$\operatorname{dom} R \stackrel{\text{def}}{=} \{\, x \mid \text{there exits } y \text{ such as } xRy \,\} \qquad \text{(\textbf{domain} of } R\text{)}$$

$$\operatorname{ran} R \stackrel{\text{def}}{=} \{\, y \mid \text{there exits } x \text{ such as } xRy \,\} \qquad \text{(\textbf{range} of } R\text{)}$$

$$\operatorname{field} R \stackrel{\text{def}}{=} \operatorname{dom} R \cup \operatorname{ran} R \qquad \text{(\textbf{field} of } R\text{)}$$

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$$\operatorname{field} R \overset{\text{def}}{=} \operatorname{dom} R \cup \operatorname{ran} R \qquad \text{(}\textbf{field}\text{ of } R\text{)}$$

If $\operatorname{field} R \subseteq X$ then R **is a relation in** X.

Binary Relations

Question

Let R be a binary relation. Are $\operatorname{dom} R$ and $\operatorname{ran} R$ sets?

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Answer

Definition of $\operatorname{dom} R$ and $\operatorname{ran} R$ without using the textbook conventions.

$$\operatorname{dom} R \stackrel{\text{def}}{=} \left\{ x \in \bigcup\bigcup R \;\middle|\; \text{there exits } y \text{ such as } xRy \right\},$$
$$\operatorname{ran} R \stackrel{\text{def}}{=} \left\{ y \in \bigcup\bigcup R \;\middle|\; \text{there exits } x \text{ such as } xRy \right\}.$$

That is, $\operatorname{dom} R$ and $\operatorname{ran} R$ are sets defined from the Axiom Scheme of Comprehension.

Convention (p. 21)

When defining sets of ordered pairs we shall use the following convention:

$$\{\, (x, y) \mid \mathbf{P}(x, y) \,\} \overset{\text{def}}{=} \{\, w \mid w \doteq (x, y) \text{ for some } x \text{ and } y \text{ such that } \mathbf{P}(x, y) \,\}.$$

Relations

Definition [2.]2.11

Let A be a set.

- The **membership relation** on A, denoted $\in_A$, is defined by

$$\in_A \stackrel{\text{def}}{=} \{\, (a,b) \mid a \in A, b \in A \text{ and } a \in b \,\}.$$

- The **identity relation** on A, denoted Id_A, is defined by

$$\mathrm{Id}_A \stackrel{\text{def}}{=} \{\, (a,b) \mid a \in A, b \in A \text{ and } a \doteq b \,\}.$$

Outline

Cartesian Products

Definition [2.]2.12

Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is defined by

$$A \times B \stackrel{\text{def}}{=} \{ (a, b) \mid a \in A \text{ and } b \in B \}.$$

Cartesian Products

Question

Let A and B be sets. Is $A \times B$ a set?

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Let A and B be sets. Is $A \times B$ a set?

Answer

Definition of $A \times B$ without using the textbook conventions.

$$A \times B \stackrel{\text{def}}{=} \{\, w \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid w \doteq (a, b) \text{ for some } a \text{ and } b \text{ such that } \mathbf{P}(a, b) \,\},$$

where $\mathbf{P}(x, y)$: "$x \in A$ and $y \in B$".

That is, $A \times B$ is a set defined from the Axiom Scheme of Comprehension.

Cartesian Products

Notation

$A^2 \stackrel{\text{def}}{=} A \times A$.

Cartesian Products

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$A^2 \stackrel{\text{def}}{=} A \times A$.

Remark

Note that a binary relation R is in A if and only if $R \subseteq A^2$.

Relations

Definition (p. 22)

Let A, B and C be sets. The **Cartesian product** of A, B and C, denoted $A \times B \times C$, is defined by

$$\begin{aligned} A \times B \times C &\overset{\text{def}}{=} (A \times B) \times C \\ &\doteq \{\, (a, b, c) \mid a \in A, b \in B \text{ and } c \in C \,\}. \end{aligned}$$

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Notation

$A^3 \stackrel{\text{def}}{=} A \times A \times A$.

Cartesian Products

Definition [2.]2.13

A set R is a **ternary relation** if all elements of R are ordered triples.

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Cartesian Products

Definition [2.]2.13

A set R is a **ternary relation** if all elements of R are ordered triples.[†]

A set R is a **ternary relation in** A if and only if $R \subseteq A^3$.

[†] See the corrections to the textbook.

Cartesian Products

Definition (p. 22)

Let A be a set. A **unary relation in** A is any subset of A.

Outline

References

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 2).