CM0832 - MT5001 Elements of Set Theory
2. Relations, Functions and Orderings

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Semester 2026-1

Outline

Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

Introduction

Starting...

> *"We begin our program of developing set theory as a foundation for mathematics by showing how various general mathematical concepts, such as relations, functions, and orderings can be represented by sets." (p. 17)*

Outline

Ordered Pairs

Definition [2.]1.1

Let a and b be sets. We define the **ordered pair** of a and b, denoted (a, b), where a is the **first coordinate** and b is the **second coordinate**, using Kuratowski (1921)'s definition, that is,

$$(a, b) \stackrel{\text{def}}{=} \{\{a\}, \{a, b\}\}.$$

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Question

Is (a, b) a set?

Ordered Pairs

Example

We show that $(\emptyset, \{\emptyset\}) \doteq\!\!\!\!\!/\ (\{\emptyset\}, \emptyset)$.

$$(\emptyset, \{\emptyset\}) \stackrel{\text{def}}{=} \{\,\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\,\} \not\doteq \{\,\{\{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\,\} \stackrel{\text{def}}{=} (\{\emptyset\}, \emptyset).$$

Ordered Pairs

Example

Let a be a set. Then

$$
\begin{aligned}
(a, a) &\stackrel{\text{def}}{=} \{\{a\}, \{a, a\}\} \\
&\doteq \{\{a\}, \{a\}\} \\
&\doteq \{\{a\}\}.
\end{aligned}
$$

Ordered Pairs

Exercise [2.]1.1

Let a and b sets.

(i) Prove that $(a, b) \in \mathcal{P}(\mathcal{P}(\{a, b\}))$.

(ii) Prove that $a, b \in \bigcup(a, b)$.

(iii) Prove that if $a \in A$ and $b \in A$, then $(a, b) \in \mathcal{P}(\mathcal{P}(A))$.

Ordered Pairs

Theorem [2.]1.8

Let a, a', b, b' be sets. Then,

$$(a, b) \doteq (a', b') \quad \text{if and only if} \quad a \doteq a' \text{ and } b \doteq b'.$$

Ordered Pairs

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Exercise

To give a different definition of ordered pair. Note that your definition must be a set and it must satisfies Theorem [2.]1.8.

Ordered Pairs

Definitions

Let a, b, c and d sets. Then

$$(a, b, c) \stackrel{\text{def}}{=} ((a, b), c) \qquad \textbf{(ordered triples)}$$

$$(a, b, c, d) \stackrel{\text{def}}{=} ((a, b, c), d) \qquad \textbf{(ordered quadruples)}$$

$$(a) \stackrel{\text{def}}{=} a \qquad \textbf{(one-tuples)}$$

Outline

Relations

Definition [2.]2.1

A set R is a **binary relation** if and only if all elements of R are ordered pairs.

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Remark

> *"A binary relation is, therefore, determined by specifying all ordered pairs of objects in that relation; it does not matter by what property the set of these ordered pairs is described." (p. 19)*

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> *"A binary relation is, therefore, determined by specifying all ordered pairs of objects in that relation; it does not matter by what property the set of these ordered pairs is described." (p. 19)*

Notation

Let R be a binary relation. We write $(a, b) \in R$ or $a \; R \; b$.

Relations

Example (informal)

Whiteboard.

Relations

Example (informal)

Whiteboard.

Example (informal)

Let $\mathbf{N} = \{0, 1, 2, \dots\}$. The identity relation on ω is defined by

$$\begin{aligned} \mathrm{Id}_\omega &= \{\, (n, n) \mid n \in \mathbf{N} \,\} \\ &= \{(0,0), (1,1), (2,2), \dots\}. \end{aligned}$$

Relations

Definition [2.]2.3

Let R be a binary relation. Then

$$\operatorname{dom} R \stackrel{\text{def}}{=} \{\, x \mid \text{there exits } y \text{ such as } x \; R \; y \,\} \qquad \text{(\textbf{domain} of } R\text{)}$$

$$\operatorname{ran} R \stackrel{\text{def}}{=} \{\, y \mid \text{there exits } x \text{ such as } x \; R \; y \,\} \qquad \text{(\textbf{range} of } R\text{)}$$

$$\operatorname{field} R \stackrel{\text{def}}{=} \operatorname{dom} R \cup \operatorname{ran} R \qquad \text{(\textbf{field} of } R\text{)}$$

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$$\operatorname{field} R \stackrel{\text{def}}{=} \operatorname{dom} R \cup \operatorname{ran} R \qquad \text{(\textbf{field} of } R\text{)}$$

If $\operatorname{field} R \subseteq X$ then R **is a relation in** X.

Relations

Question

Let R be a binary relation. Are $\operatorname{dom} R$ and $\operatorname{ran} R$ sets?

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Answer

Definition of $\operatorname{dom} R$ and $\operatorname{ran} R$ without using the textbook conventions.

$$\operatorname{dom} R \stackrel{\text{def}}{=} \left\{ x \in \bigcup\bigcup R \;\middle|\; \text{there exits } y \text{ such as } x \; R \; y \right\},$$
$$\operatorname{ran} R \stackrel{\text{def}}{=} \left\{ y \in \bigcup\bigcup R \;\middle|\; \text{there exits } x \text{ such as } x \; R \; y \right\}.$$

That is, $\operatorname{dom} R$ and $\operatorname{ran} R$ are sets defined from the Axiom Scheme of Comprehension.

Relations

Convention (p. 21)

When defining sets of ordered pairs we shall use the following convention:

$$\{\,(x,y) \mid \mathbf{P}(x,y)\,\} \stackrel{\text{def}}{=} \{\,w \mid w \doteq (x,y) \text{ for some } x \text{ and } y \text{ such that } \mathbf{P}(x,y)\,\}.$$

Relations

Definition [2.]2.11

Let A be a set.

- The **membership relation** on A, denoted by $\in_A$ is defined by

$$\in_A \stackrel{\text{def}}{=} \{ (a, b) \mid a \in A, b \in A \text{ and } a \in b \}.$$

- The **identity relation** on A, denoted by Id_A, is defined by

$$\mathrm{Id}_A \stackrel{\text{def}}{=} \{ (a, b) \mid a \in A, b \in A \text{ and } a \doteq b \}.$$

Relations

Definition [2.]2.12

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is defined by

$$A \times B \stackrel{\text{def}}{=} \{ (a, b) \mid a \in A \text{ and } b \in B \}.$$

Relations

Question

Let A and B be sets. Is $A \times B$ a set?

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Answer

Definition of $A \times B$ without using the textbook conventions.

$$A \times B \stackrel{\text{def}}{=} \{\, w \in \mathcal{P}(\mathcal{P}((A \cup B))) \mid w \doteq (a, b) \text{ for some } a \text{ and } b \text{ such that } \mathbf{P}(a, b) \,\},$$

where $\mathbf{P}(x, y)$: "$x \in A$ and $y \in B$".

That is, $A \times B$ is a set defined from the Axiom Scheme of Comprehension.

Relations

Notation

$A^2 \stackrel{\text{def}}{=} A \times A$.

Relations

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Remark

Note that a binary relation R is in A if and only if $R \subseteq A^2$.

Relations

Definition (p. 22)

Let A, B and C be sets. The **Cartesian product** of A, B and C, denoted by $A \times B \times C$, is defined by

$$\begin{aligned} A \times B \times C &\stackrel{\text{def}}{=} (A \times B) \times C \\ &\doteq \{\, (a, b, c) \mid a \in A, b \in B \text{ and } c \in C \,\}. \end{aligned}$$

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Notation

$A^3 \stackrel{\text{def}}{=} A \times A \times A$.

Relations

Definition [2.]2.13

A set R is a **ternary relation** if and only if all elements of R are ordered triples.

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A set R is a **ternary relation in** A if and only if $R \subseteq A^3$.

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Definition [2.]2.13

A set R is a **ternary relation** if and only if all elements of R are ordered triples.[†]

A set R is a **ternary relation in** A if and only if $R \subseteq A^3$.

[†]See the corrections to the textbook.

Relations

Definition (p. 22)

Let A be a set. A **unary relation in A** is any subset of A.

Outline

Functions

Definition [2.]3.1

A binary relation F is a **function** (**mapping** or **correspondence**) if and only if

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for each a in $\operatorname{dom} F$ there is exactly one b such that $a\ F\ b$.

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for each a in $\operatorname{dom} F$ there is exactly one b such that $a \ F \ b$.

Example

Whiteboard.

Functions

Notation (p. 24)

If F is a function with $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$ we shall write $F : A \to B$, $\langle F(a) \mid a \in A \rangle$ or $\langle F_a \rangle_{a \in A}$.

Functions

Discussion

Quote from the textbook:

> *"Function, as understood in mathematics, is a procedure, a rule, assigning to any object a from the domain of the function a unique object b, the value of the function at a." (p. 28)*

Is there any (implicit or explicit) procedure, rule or formula in our definition of function?

Functions

Remark

The current definition of a (one-value) function (of a real variable) is from Dirichlet who in 1837 wrote:

> *"If a variable y is so related to a variable x that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a **function** of the independent variable x."*
> *(Merzcbach and Boyer [1968] 2011, p. 452).*

Functions

Remark

The current definition of a function on arbitrary sets if from Cantor 1895:

> *"By a **covering of the aggregate** N **with elements of the aggregate**, M or, more simply, by a **covering of** N **with** M, we understand a law by which with every element n of N a definite element of M is bound up, where one and the same element of M can come repeatedly into application. The element of M bound up with n is, in a way, a one-valued function of n, and may be denoted by $f(n)$; it is called a **covering function of** n. The corresponding covering of N will be called $f(N)$." (Cantor [1895] 1915, p. 94)*

Functions

Lemma [2.]3.2 (function extensionality)

Let F and G be functions. $F = G$ if and only if $\operatorname{dom} F = \operatorname{dom} G$ and $F(x) = G(x)$, for all $x \in \operatorname{dom} F$.

Functions

Definitions [2.]3.3

Let F be a function and A and B sets.

(i) F is a function **on** (**from**) A if and only if $\operatorname{dom} F = A$.

(ii) F is a function **into** (**to**) B if and only if $\operatorname{ran} F \subseteq B$.

(iii) F is a function **onto** B if and only if $\operatorname{ran} F = B$.

(iv) The **restriction** of F to A, denoted by $F \restriction A$, is the function defined by

$$F \restriction A \stackrel{\text{def}}{=} \{ (x, y) \in F \mid x \in A \}.$$

Functions

Definition [2.]3.7

A function F is **one-to-one** (or **injective**) if and only if for each $y \in \operatorname{ran} F$ there is only one x such that $x\ F\ y$. In other words, if $x_1, x_2 \in \operatorname{dom} F$ and $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

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Example

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Definition

A function F is an **one-to-one correspondence** between A and B if and only if F is an one-to-one function from A onto B.

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Let A and B be sets. We define the **set of functions** from A into B, denoted B^A.

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- $\{0,1\}^{\mathbf{N}}$: The set of infinity binary sequences.

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- $\emptyset^A = \emptyset$ for $A \neq \emptyset$ (no function can have a non-empty domain and an empty range).
- $A^\emptyset = \{\emptyset\}$ for any set A ($\emptyset$ is the only function with an empty domain).

Functions

Question

Let A and B be sets. Is B^A a set?

Functions

Question

Let A and B be sets. Is B^A a set?

Answer

Definition of B^A without using the textbook conventions.

$$B^A \stackrel{\text{def}}{=} \{ F \in \mathcal{P}((A \times B)) \mid F : A \to B \}.$$

That is, B^A is a set defined from the Axiom Scheme of Comprehension.

Outline

References

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