CM0832 - MT5001 Elements of Set Theory
Properties of Natural Numbers

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Semester 2026-1

Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

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Introduction

The Induction Principle

Linear Ordering on $\mathbb{N}$

The Induction Principle, Second Version

Well-Ordering on $\mathbb{N}$

References

Introduction

Example (informal)

Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Introduction

Let us remember that the set of natural numbers $\mathbf{N}$ is the smallest set such that

(a) $0 \in \mathbf{N}$,

(b) if $n \in \mathbf{N}$ then $n + 1 \in \mathbf{N}$,

where

$0 \stackrel{\text{def}}{=} \emptyset$ and

$n + 1 \stackrel{\text{def}}{=} \mathrm{S}(n) \stackrel{\text{def}}{=} n \cup \{ n \}$.

Outline

The Induction Principle

The Induction Principle (mathematical induction) (p. 42)

Let $\mathbf{P}(x)$ be a property. If

(a) $\mathbf{P}(0)$ holds,

(b) for all $n \in \mathbf{N}$, $\mathbf{P}(n)$ implies $\mathbf{P}(n+1)$,

then the property $\mathbf{P}$ holds for all natural numbers n.

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Proof

(a) From hypotheses (a) and (b) we know that the set $A = \{\, n \in \mathbf{N} \mid \mathbf{P}(n) \,\}$ is inductive.

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Proof

(a) From hypotheses (a) and (b) we know that the set $A = \{\, n \in \mathbf{N} \mid \mathbf{P}(n) \,\}$ is inductive.

(b) Since $\mathbf{N}$ is the smallest inductive set then $\mathbf{N} \subseteq A$. That is, $\mathbf{P}(n)$ holds for all $n \in \mathbf{N}$.

The Induction Principle

Remark

Recall that $m < n \stackrel{\text{def}}{=} m \in n$, for all $m, n \in \mathbf{N}$.

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Lemma [3.]2.1

(a) $0 \leq n$ for all $n \in \mathbf{N}$.

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Linear Ordering on $\mathbf{N}$

Theorem [3.]2.2

$(\mathbf{N}, <)$ is a linearly ordered set.

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The Induction Principle, Second Version

Induction Principle, Second Version (strong induction) (p. 44)

Let $\mathbf{P}(x)$ be a property. Assume that, for all $n \in \mathbf{N}$,

$$\text{if } \mathbf{P}(k) \text{ holds for all } k < n, \text{ then } \mathbf{P}(n).$$

Then $\mathbf{P}$ holds for all natural numbers n.

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Well-Ordered Sets

Definition [3.]2.3

A linear ordering $\prec$ of a set A is a **well-ordering** if every non-empty subset of A has a least element.

The ordered set $(A, \prec)$ is a **well-ordered set**.

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Remark

"*Well-ordered sets form a backbone of set theory.*" *(p. 45)*

Well-Ordered Sets

Theorem [3.]2.4

$(\mathbf{N}, <)$ is a well-ordered set.

Outline

References

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 2).