CM0832 - MT5001 Elements of Set Theory
Orderings

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Semester 2026-1

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

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Introduction

Description

We can use relations to order some or all the elements of a set.

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Examples (some order relations)

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Examples (some order relations)

- The words in a dictionary

  aRb if a comes before b in the dictionary.
- Academic genealogical descent

  aRb if a was the supervisor of the thesis of b.
- Schedule projects

  aRb if a is a task that must be completed before the task b begins.

Introduction

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- What means that a binary relation R is an ordering relation on a set A?

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- Orderings are binary relations.
- What means that a binary relation R is an ordering relation on a set A?
- Non-strict orderings $(\preccurlyeq)$ and strict orderings $(\prec)$ can be interchangeably defined.

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Antisymmetric Relations

Definition [2.]5.1

A binary relation R in a set A is **antisymmetric in** A if
for all $x, y \in A$, xRy and yRx imply $x \doteq y$.

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Examples

Whiteboard.

Non-Strict Partial Orderings

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(a) The adjective "non-strict" means that for all $a \in P$, $a \preccurlyeq a$.

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(a) The adjective "non-strict" means that for all $a \in P$, $a \preccurlyeq a$.

(b) The adjective "partial" means that not every pair of elements of P needs to be comparable, i.e., there may be $a, b \in P$, such that, $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$.

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(c) The textbook uses the terms "partial ordering" or "ordering" instead of the term "non-strict partial ordering".

Non-Strict Partial Orderings

Examples

(a) The relation $\leq$ is a non-strict partial ordering on the set of all (natural, rational, real) numbers.

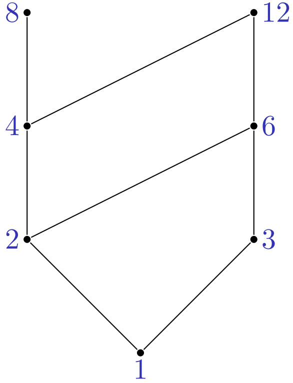
(b) Let $\subseteq_A$ a binary relation in A defined by: $x \subseteq_A y$ if and only if $x \subseteq y$ and $x, y \in A$. The relation $\subseteq_A$ is a non-strict partial ordering of the set A.

(c) The relation Id_A is a non-strict partial ordering of A.

Hasse Diagrams

Example (informal)

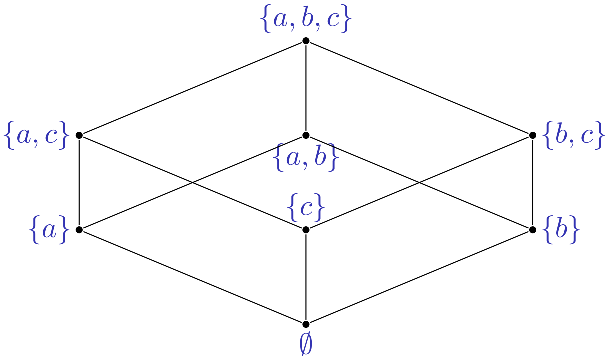
Let $|$ be the relation of divisibility. Hasse diagram for the non-strictly poset $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.



Hasse Diagrams

Example (informal)

Let $A = \{a, b, c\}$. Hasse diagram for the non-strictly poset $(\mathcal{P}(A), \subseteq_{\mathcal{P}(A)})$.



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Asymmetric Relations

Definition [2.]5.4

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(a) The adjective "strict" means that for all $a \in P$, $a \not\prec a$.

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(c) The textbook uses the term "strict ordering" instead of the term "strict partial ordering".

Strict Partial Orderings

Examples

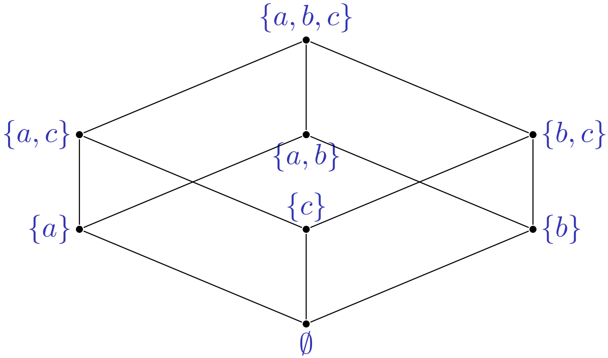
(a) The relation $<$ is a strict partial ordering on the set of all (natural, rational, real) numbers.

(b) Let $\subset_A$ (proper subset) a binary relation in A defined by: $x \subset_A y$ if and only if $x \subset y$ and $x, y \in A$. The relation $\subseteq_A$ is a non-strict partial ordering of the set A.

Hasse Diagrams

Example (informal)

Let $A = \{a, b, c\}$. Hasse diagram for the strictly poset $(\,\mathcal{P}(A), \subset_{\mathcal{P}(A)}\,)$.



Partial Orderings

Theorem [2.]5.6 (relationship between non-strict and strict partial orderings)

(a) Let $\preccurlyeq$ be a non-strict partial ordering on a set P. The relation

$$x \prec y \overset{\text{def}}{=} x \preccurlyeq y \text{ and } x \mathrel{\dot{\neq}} y$$

is a strict partial ordering on P.

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is a strict partial ordering on P.

(b) Let $\prec$ be a strict partial ordering on a set P. The relation

$$x \preccurlyeq y \overset{\text{def}}{=} x \prec y \text{ or } x \doteq y$$

is a non-strict partial ordering on P.

Outline

Comparable Elements

Definition [2.]5.7

(a) Let $\preccurlyeq$ be a non-strict partial ordering of P. Two elements $a, b \in P$ are **comparable** in the ordering $\preccurlyeq$ if

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$$a \doteq b \quad \text{or} \quad a \prec b \quad \text{or} \quad b \prec a.$$

(c) Two elements $a, b \in P$ are **incomparable** if they are not comparable.

Linear Orderings

Conventions

(a) The symbol $\preccurlyeq$ shall denote a non-strict partial order.

(b) The symbol $\prec$ shall denote a strict partial order.

Linear Orderings

Definition [2.]5.9

A binary relation R in a set L is a **linear ordering** (or **total ordering**) if R is a partial ordering of L and any two elements of L are comparable.

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A binary relation R in a set L is a **linear ordering** (or **total ordering**) if R is a partial ordering of L and any two elements of L are comparable.

The pair (L, R) is a **linearly ordered set** (or **totally ordered set**).

Linear Orderings

Examples

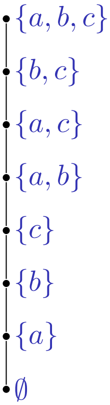
(a) The relations $\leq$ and $<$ are linear orderings on the set of all (natural, rational, real) numbers.

(b) Neither the relation $\subseteq_A$ nor the relation $\subset_A$ are total orderings of A.

Hasse Diagrams

Example (informal)

Let $A = \{a, b, c\}$. A linearly ordered set $(\,\mathcal{P}(A), \preccurlyeq\,)$ represented by its Hasse diagram.



Example (informal)

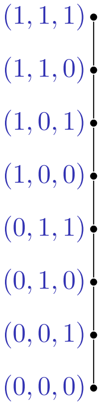
Let's define all the lineal non-strict orderings on the set $\{1, 2, 3\}$.

(a) (b) (c) (d) (e) (f)

Hasse Diagrams

Example (informal)

Let $A = \{0, 1\}$. A linearly ordered set $(\, A \times A \times A, \prec \,)$ represented by it Hasse diagram.



Outline

Notable Elements

Definition [2.]5.11

Let $\preccurlyeq$ be an ordering of P, and let $A \subseteq P$.

(a) An element $a \in A$ is the **least** element of A in the ordering $\preccurlyeq$ if $a \preccurlyeq x$ for every $x \in A$.

(b) An element $a \in A$ is a **minimal** element of A in the ordering $\preccurlyeq$ if there exists no $x \in A$ such that $x \preccurlyeq a$ and $x \doteq\!\!\!\!/\; a$.

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Similarly,

(a') An element $a \in A$ is the **greatest** element of A in the ordering $\preccurlyeq$ if $x \preccurlyeq a$ for every $x \in A$.

(b') An element $a \in A$ is a **maximal** element of A in the ordering $\preccurlyeq$ if there exists no $x \in A$ such that $a \preccurlyeq x$ and $x \doteq\!\!\!\!/\; a$.

Notable Elements

Example (informal, $A = P$)

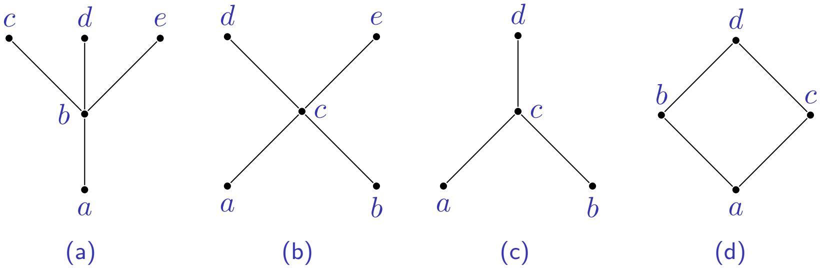

Fig.	Least element of A	Greatest element of A	Maximals of A	Minimals of A
(a)	a		c, d, e	a
(b)			d, e	a, b
(c)		d	d	a, b
(d)	a	d	d	a

Definition [2.]5.14

Let $\preccurlyeq$ be an ordering of P, and let $A \subseteq P$.

(a) An element $p \in P$ is a **lower bound** of A in the ordered set $(\,P, \preccurlyeq\,)$ if $p \preccurlyeq x$ for all $x \in A$.

(b) An element $p \in P$ is the **infimum** (or **greatest lower bound**) of A in the ordered set $(\,P, \preccurlyeq\,)$, denoted $\inf(A)$, if it is the greatest element of the set of all lower bounds of A in $(\,P, \preccurlyeq\,)$.

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Similarly,

(a') An element $p \in P$ is an **upper bound** of A in the ordered set $(P, \preccurlyeq)$ if $x \preccurlyeq p$ for all $x \in A$.

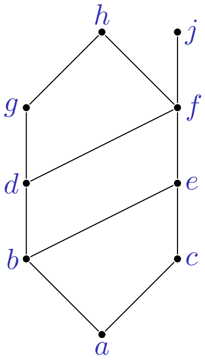
(b') An element $p \in P$ is the **supremun** (or **least upper bound**) of A in the ordered set $(P, \preccurlyeq)$, denoted $\sup(A)$, if it is the least element of the set of all upper bounds of A in $(P, \preccurlyeq)$.

Notable Elements

Example (informal)

- $A = \{a, b, c\}$
  Upper bounds: $\{e, f, j, h\}$
  $\sup(A) = e$
  Lower bounds: $\{a\}$
  $\inf(A) = a$

- $A = \{j, h\}$
  No upper bounds.
  Lower bounds: $\{a, b, c, d, e, f\}$
  $\inf(A) = f$



- $A = \{a, c, d, f\}$
  Upper bounds: $\{f, h, j\}$
  $\sup(A) = f$
  Lower bounds: $\{a\}$
  $\inf(A) = a$

- $A = \{b, d, g\}$
  Upper bounds: $\{g, h\}$
  $\sup(A) = g$
  Lower bounds: $\{a, b\}$
  $\inf(A) = b$

Outline

References

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 3).