CM0832 - MT5001 Elements of Set Theory
Introduction

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Universidad EAFIT

Semester 2026-1

Pacto pedagógico

Como miembros de la Universidad EAFIT, nos comprometemos a actuar de manera íntegra siguiendo los más altos estándares éticos y morales.

- Respeto
- Tolerancia
- Honradez
- Compromiso

Pacto pedagógico

Página web del curso

https://asr.github.io/courses/cm0832-set-theory/2026-1/

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Conducto regular, fechas y porcentajes de las evaluaciones

La información está en la página web del curso.

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Responsabilidad compartida

- Profesor
- Estudiantes

Pacto pedagógico

Asistencia a clase

Reglamento académico de los programas de posgrado, Capítulo VI, Artículo 62, Parágrafo 2.

> *"El estudiante de posgrado cuyas faltas de asistencia lleguen al treinta por ciento (30%) del total de las horas de clase programadas para el curso o para una parte de éste, cuando se desarrolle con más de un profesor, en secciones temáticas denominadas 'módulo', pierde con calificación de cero punto cero (0.0) del seminario o curso correspondiente y esta nota afecta el promedio crédito acumulado."*

Pacto pedagógico

Orientaciones para el curso

- Se recomienda cuatro horas de trabajo por semana (dos horas por cada hora de clase).
- Las clases son presenciales.
- La evaluación a la docencia es obligatoria.
- Se recomienda revisar periódicamente los canales de comunicación institucionales (EAFIT Interactiva y el correo institucional).
- El estudiante podrá entrar a clase a más tardar 20 minutos después de la hora asignada para su inicio.

Course Outline

- Formal languages and theories. Methods of proof. Naive set theory.
- Axiomatic set theory. Operations on sets.
- Relations, functions and orderings.
- Natural numbers.
- Finite, countable and uncountable sets.
- Cardinal numbers.
- Ordinal numbers.
- Axiom of choice.

Preliminaries

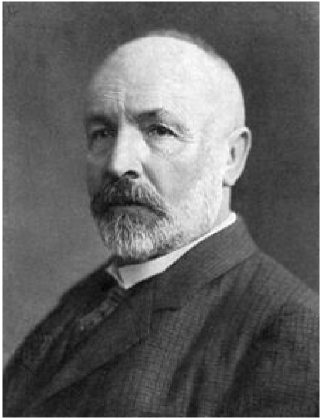
Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Origins

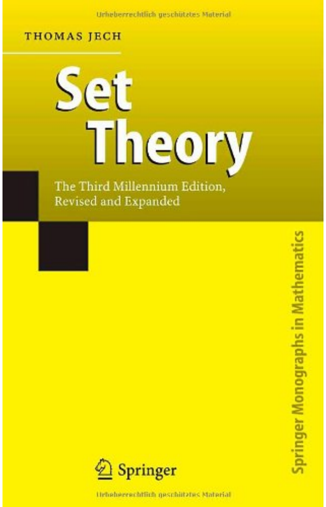


Georg Cantor (1845 – 1918)†

Cantor around 1870

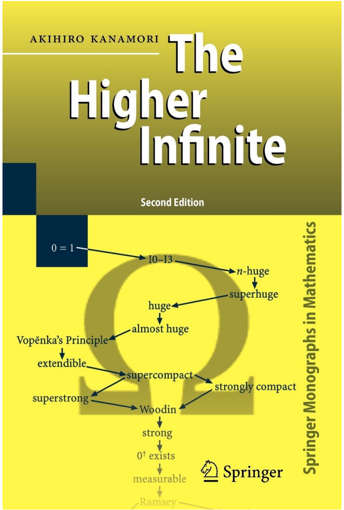
†Figures source: https://en.wikipedia.org/wiki/Georg_Cantor .

Origins



"Set theory was invented by Georg Cantor... It was however Cantor who realized the significance of one-to-one functions between sets and introduced the notion of cardinality of a set." (Jech [1978] 2006, p. 15)

Origins



"Set theory was born on that December 1873 day when Cantor established that the reals are uncountable, i.e. there is no one-to-one correspondence between the reals and the natural numbers." (Kanamori [1994] 2009, p. XII)

Naive Set Theory

Remark

Cantor set theory is also called naive set theory.

Naive Set Theory

Cantor's set definition

*"Unter einer **Menge** verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschauung oder unseres Denkens (welche die **Elemente** von M genannt werden) zu einem Ganzen."*
(Cantor 1895, p. 481)

*"By an **aggregate** (Menge) we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought. These objects are called the **elements** of M." (Cantor [1915] 1955, p. 85)*

*"A **set** is a collection into a whole of definite, distinct objects of our intuition or our thought. The objects are called **elements (members)** of the set."*
(Hrbacek and Jech [1978] 1999, p. 1)

Naive Set Theory

Description

*"**Sets** are not objects of the real world, like tables or stars; they are created by our mind, not by our hands.... The human mind possesses the ability to abstract, to think of a variety of different objects as being bound together by some common property, and thus to form a set of objects having that property."*
(Hrbacek and Jech [1978] 1999, p. 1)

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Examples

In the lecture.

Naive Set Theory

Remark

A classical text in naive set theory is (Halmos 1960).

Naive Set Theory

Problem

Implicit use of “theorems” on sets.

† Figure source: https://commons.wikimedia.org/w/index.php?curid=48219447 .

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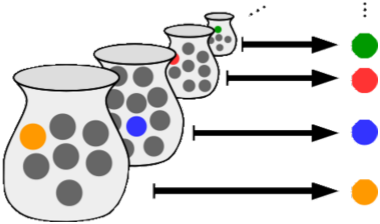
An example of such "theorem" was the use of axiom of choice illustrated by the figure.[†]



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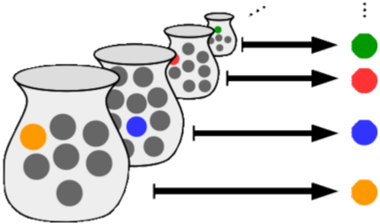

A very complete history of origins, development and influence of the axiom of choice is in (Moore 1982).

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Naive Set Theory

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What is a unary property?

This problem was the cause of Russell's paradox: Let R be a "set" defined by

$$R = \{ x \mid x \notin x \}.$$

The Crisis in the Foundations of Mathematics

$$\text{Paradoxes} \Rightarrow \text{Crisis} \Rightarrow \begin{cases} \text{Logicism} & \text{(Russell and Whitehead)} \\ \text{Formalism} & \text{(Hilbert)} \\ \text{Intuitionism} & \text{(Brouwer)} \end{cases}$$

The Crisis in the Foundations of Mathematics

Logicism (Russell and Whitehead)

> *"The logicistic thesis is that mathematics is a branch of logic. The mathematical notions are to be defined in terms of the logical notions. The theorems of mathematics are to be proved as theorems of logic." (Kleene [1952] 1974, p. 43)*

The Crisis in the Foundations of Mathematics

Formalism (Hilbert)

> *"Classical mathematics shall be formulated as a formal axiomatic theory, and this theory shall be proved to be consistent, i.e. free from contradiction."*
> *(Kleene [1952] 1974, p. 53)*

The Crisis in the Foundations of Mathematics

Intuitionism (Brouwer)

> *"Intuitionism is based on the idea that mathematics is a creation of the mind. The truth of a mathematical statement can only be conceived via a mental construction that proves it to be true." (Iemhoff 2024)*

Axiomatic Set Theory

Description

"We formulate some of the relatively simple properties of sets used by mathematicians as axioms, and then take care to check that all theorems follow logically from the axioms. Since the axioms are obviously true and the theorems logically follow from them, the theorems are also true (not necessarily obviously). We end up with a body of truths about sets which includes, among other things, the basic properties of natural, rational, and real numbers, functions, orderings, etc., but as far as is known, no contradictions." (Hrbacek and Jech [1978] 1999, p. 3)

Axiomatic Set Theory

Axiomatic set theory as a fundational system for mathematics

- *"Our axioms provide a sufficient collection of assumptions for the development of the whole of mathematics—a remarkable fact." (Enderton 1977, p. 11)*
- *"Experience has shown that practically all notions used in contemporary mathematics can be defined, and their mathematical properties derived, in this axiomatic system. In this sense, the axiomatic set theory serves as a satisfactory foundations for the other branches of mathematics." (Hrbacek and Jech [1978] 1999, p. 3)*
- *"But why axiomatize set theory in the first place? Well, for one thing, it is well known that set theory provides a unified framework for the whole of pure mathematics, and surely if anything deserves to be put on a sound basis it is such a foundational subject." (Devlin [1979] 1993, p. 29)*
- *"Conventional mathematics is based on ZFC (the Zermelo-Fraenkel axioms, including the Axiom of Choice). Working withing ZFC, on develops:...All the mathematics found in basic texts on analysis, topology, algebra, etc." (Kunen [2011] 2013, p. 1)*

Axiomatic Set Theory

Some axiomatic systems of set theory

- Zermelo-Fraenkel set theory (ZF)
- Zermelo-Fraenkel set theory with Choice (ZFC)
- von Neumann-Bernays-Gödel set theory (NBG)
- Morse-Kelley set theory (MK)
- Tarski-Grothendieck set theory (TG)

Axiomatic Set Theory

But!

> *"On the other hand, we do not claim that every true fact about sets can be derived from the axioms we present. The axiomatic system is not complete in this sense." (Hrbacek and Jech [1978] 1999, p. 3)*

Foundations of Mathematics

Foundational systems

(i) Set theories (with additional axioms)
(ii) Category theories
(iii) Type theories
(iv) Univalent foundations
(v) Homotopy type theories

Foundations of Mathematics

Foundational systems[†]

(i) Set theories (with additional axioms)
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[†] See, e.g. (Centrone, Kant and Sarikaya 2019).

First-order logic: Two historical remarks

> *"First-order logic was explicitly identified by Peirce in 1885, but then forgotten. It was independently re-discovered in Hilbert's 1917/18 lectures, and given wide currency in the 1928 monograph, Hilbert & Ackermann. Peirce was the first to identify it: but it was Hilbert who put the system on the map."* (Ewald 2019)

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> *"Nevertheless, Hilbert did not at any point regard first-order logic as the proper basis for mathematics... It was in Skolem's work on set theory (1923) that first-order logic was first proposed as all of logic and that set theory was first formulated within first-order logic." (Moore 1988, p. 128)*

First-Order Theories

Notation: Logical constants

$\wedge$	(and)	conjunction
$\vee$	(or)	inclusive[†] disjunction
$\rightarrow$	(if __, then __)	conditional, material implication
$\neg$	(not)	negation
$\leftrightarrow$	(if and only if)	bi-conditional, material equivalence
$\bot$	(falsity)	bottom, falsum
$(\forall x)$	(for every x)	universal quantifier
$(\exists x)$	(there exists a x)	existential quantifier
$(\exists! x)$	(there exists one and only one x)	unique existential quantifier
$\doteq$	(equal)	identity, equality

[†]One or the other or both.

First-Order Theories

Preliminaries logics

- First-order logic with identity
- Non-logic symbols and non-logic axioms
- Theories
- Definitions

First-Order Theories

Description

> *"The adjective 'first-order' is used to distinguish the languages we shall study here from those in which there are predicates having other predicates or functions as arguments or in which predicate quantifiers or function quantifiers are permitted, or both." (Mendelson [1964] 2015, p. 53)*

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Remark

For an introduction to first-order languages and first-order theories, see e.g. (Hamilton 1978) or (Mendelson [1964] 2015).

First-Order Theories

Example (first-order Dedekind-Peano arithmetic)

- Non-logical symbols

  The formal language $\mathfrak{L}$ of the first-order theory of arithmetic (FA) is defined by

$$\mathfrak{L} = \{', +, *, 0\}, \quad \text{where}$$

(i) the symbol $'$ is an unary function symbol (successor function),
(ii) the symbol $+$ is a binary function symbol (addition function),
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That is, the symbols $\{', +, *, 0\}$ are primitive (undefined) concepts in FA.

(continued on next slide)

First-Order Theories

Example (first-order Dedekind-Peano arithmetic)

- Non-logical axioms of FA

$$(\forall n)(0 \not\doteq n') \qquad (\mathrm{FA}_1)$$

$$(\forall m)(\forall n)(m' \doteq n' \to m \doteq n) \qquad (\mathrm{FA}_2)$$

$$(\forall n)(n + 0 \doteq n) \qquad (\mathrm{FA}_3)$$

$$(\forall m)(\forall n)(m + n' \doteq (m + n)') \qquad (\mathrm{FA}_4)$$

$$(\forall n)(n * 0 \doteq 0) \qquad (\mathrm{FA}_5)$$

$$(\forall m)(\forall n)(m * n' \doteq (m * n) + m) \qquad (\mathrm{FA}_6)$$

For any unary property P,

$$[P(0) \wedge (\forall n)(P(n) \to P(n'))] \to (\forall n)(Pn) \qquad (\mathrm{FA}_7)$$ (axiom schema of induction)

First-Order Theories

Example (first-order Dedekind-Peano arithmetic)

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† See, e.g. (Machover 1996; Hájek and Pudlák [1993] 1998; Skolem 1955; Robinson 1949).

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