CM0832 - MT5001 Elements of Set Theory
3.1 The Recursion Theorem

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

Outline

Introduction

"Our next task is to show how to define addition, multiplication, and other familiar operations of arithmetic. To facilitate this, we develop an important general method for defining functions on $\mathbf{N}$*." (p. 46)*

Outline

Sequences

Definition (p. 46)

A **sequence** is a function whose domain is either

(a) a natural number n or

(b) the set $\mathbf{N}$.

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Definitions (p. 46)

(a) A **finite sequence of length** n is a sequence whose domain is $n \in \mathbf{N}$.

(b) An **infinite sequence** is a sequence whose domain is $\mathbf{N}$.

Notation (p. 46)

- A finite sequence of length n is denoted by

$$\langle\, a_i \mid i < n \,\rangle \quad \text{or} \quad \langle\, a_i \mid i \doteq 0, 1, \ldots, n-1 \,\rangle \quad \text{or} \quad \langle\, a_0, a_1, \ldots, a_{n-1} \,\rangle.$$

- The range of a finite sequence of length n is denoted by

$$\{\, a_i \mid i \in \mathbf{N} \,\} \quad \text{or} \quad \{\, a_0, a_1, \ldots, a_{n-1} \,\}.$$

Sequences

Notation (p. 46)

- An infinite sequence is denoted by

$$\langle\, a_i \mid i \in \mathbf{N} \,\rangle \quad \text{or} \quad \langle\, a_i \mid i \doteq 0, 1, \ldots \,\rangle \quad \text{or} \quad \langle\, a_i \,\rangle_{i \doteq 0}^{\infty} \;.$$

- The range of an infinite sequence is denoted by

$$\{\, a_i \mid i \in \mathbf{N} \,\} \quad \text{or} \quad \{\, a_i \,\}_{i \doteq 0}^{\infty} \;.$$

Sequences

Example

The **empty sequence** $\langle\ \rangle$ is the finite sequence of length 0.

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Recall that B^A denotes the set of all functions on A into B.

The set of all finite sequences of elements of A, denoted $\mathrm{Seq}(A)$, is defined by

$$\mathrm{Seq}(A) \stackrel{\text{def}}{=} \bigcup_{n \in \mathbf{N}} A^n.$$

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Example

Note that an infinite sequence of elements of A is an element of $A^{\mathbf{N}}$.

Sequences

Example

- A sequence $s : \mathbf{N} \to \mathbf{N}$ is defined by

$$\begin{aligned} s_0 &= 1, \\ s_{n+1} &= n^2, \text{ for all } n \in \mathbf{N}. \end{aligned}$$

Sequences

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$$s_{n+1} = n^2, \text{ for all } n \in \mathbf{N}.$$

- From the definition of s we could define a property $\mathbf{P}(x, y)$ such that

$$s_x \doteq y \quad \text{if and only if} \quad \mathbf{P}(x, y).$$

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- For proving that s is a sequence (existence and uniqueness) we could define a property $\mathbf{P}(x, y)$: "either $x \doteq 0$ and $y \doteq 1$ or, for some $n \in \mathbb{N}, x + 1$ and $y \doteq n^2$",

  and use the axioms on

$$s = \{\, (x, y) \in \mathbf{N} \times \mathbf{N} \mid \mathbf{P}(x, y) \,\}.$$

Sequences

Example

A sequence $f : \mathbf{N} \to \mathbf{N}$ is defined by

$$\begin{aligned} f_0 &= 1, \\ f_{n+1} &= f_n \times (n+1), \text{ for all } n \in \mathbf{N}. \end{aligned}$$

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A sequence $f : \mathbf{N} \to \mathbf{N}$ is defined by

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Question

From the definition of f how we could define a property $\mathbf{P}(x, y)$ such that

$$f_x \doteq y \quad \text{if and only if} \quad \mathbf{P}(x, y)$$

?

Outline

Recursion on Natural Numbers

Recursion theorem on $\mathbf{N}$

For any set A, any $a \in A$, and any function $g : A \times \mathbf{N} \to A$, there exists a unique infinite sequence $f : \mathbf{N} \to A$ such that

(a) $f_0 = a$,

(b) $f_{n+1} = g(f_n, n)$, for all $n \in \mathbf{N}$.

Outline

References

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 2).