

CM0832 - MT5001 Elements of Set Theory
3.2 Properties of Natural Numbers

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Semester 2026-1

Outline

Introduction

The Induction Principle

Linear and Well Orderings on \mathbb{N}

References

Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Introduction

Example (informal)

Prove that $\sum_{i=0}^n 2^i = 2^{n+1} - 1$.

Introduction

Let us remember that the set of natural numbers \mathbf{N} is the smallest set such that

(a) $0 \in \mathbf{N}$.

(b) If $n \in \mathbf{N}$ then $n + 1 \in \mathbf{N}$.

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The Induction Principle

The Induction Principle (mathematical induction) (p. 42)

Let $\mathbf{P}(x)$ be a property. If

(a) $\mathbf{P}(0)$ holds and

(b) for all $n \in \mathbf{N}$, $\mathbf{P}(n)$ implies $\mathbf{P}(n + 1)$,

then \mathbf{P} holds for all natural numbers n .

The Induction Principle

The Induction Principle (mathematical induction) (p. 42)

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(a) $\mathbf{P}(0)$ holds and

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then \mathbf{P} holds for all natural numbers n .

Proof

(a) From hypotheses (a) and (b) we know that the set $A = \{n \in \mathbf{N} \mid \mathbf{P}(n)\}$ is inductive.

(b) Since \mathbf{N} is the smallest inductive set then $\mathbf{N} \subseteq A$. That is, $\mathbf{P}(n)$ holds for all $n \in \mathbf{N}$.

The Induction Principle

Lemma [3.]2.1

- (a) $0 \leq n$ for all $n \in \mathbf{N}$.
- (b) For all $k, n \in \mathbf{N}$, $k < n + 1$ if and only if $k < n$ or $k = n$.

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Linear Ordering on \mathbf{N}

Theorem [3.]2.2

$(\mathbf{N}, <)$ is a linearly ordered set.

The Induction Principle, Second Version

Induction Principle, Second Version (p. 44)

Let $\mathbf{P}(x)$ be a property (possibly with parameters). Assume that, for all $n \in \mathbf{N}$,

(*) if $\mathbf{P}(k)$ holds for all $k < n$, then $\mathbf{P}(n)$.

Then \mathbf{P} holds for all natural numbers n .

Well-Ordered Sets

Definition [3.]2.3

A linear ordering \prec of a set A is a **well-ordering** if every non-empty subset of A has a least element.

The ordered set (A, \prec) is a **well-ordered set**.

Well-Ordered Sets

Theorem [3.]2.4

$(\mathbf{N}, <)$ is a well-ordered set.

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Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 3).