CM0832 - MT5001 Elements of Set Theory
3.1 Introduction to Natural Numbers

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Outline

Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Defining the Natural Numbers

Approaches for introducing mathematical objects

- Axiomatic
- Definitional

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Definitional approach for introducing natural numbers

- We shall define natural numbers in terms of sets.
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- We shall define natural numbers in terms of sets.
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Question

How to define natural numbers in terms of sets?

Towards an Inductive Definition

We need an inductive definition like

(a) 0 is a natural number.

(b) If n is a natural number then $n + 1$ is a natural number.

(c) All natural numbers are obtained by application of (a) and (b).

Outline

Inductive Sets

Definition [3.]1.1

The **successor** of a set a is

$$\mathrm{S}(a) \stackrel{\mathrm{def}}{=} a \cup \{ a \}.$$

Inductive Sets

Example

(a) Some successors

$$
\begin{aligned}
S(\emptyset) &\doteq \emptyset \cup \{\,\emptyset\,\} && \doteq \{\,\emptyset\,\},\\
S(\{\,\emptyset\,\}) &\doteq \{\,\emptyset\,\} \cup \{\,\{\,\emptyset\,\}\,\} && \doteq \{\,\emptyset, \{\,\emptyset\,\}\,\},\\
S(\{\,\emptyset, \{\,\emptyset\,\}\,\}) &\doteq \{\,\emptyset, \{\,\emptyset\,\}\,\} \cup \{\,\{\,\emptyset, \{\,\emptyset\,\}\,\}\,\} && \doteq \{\,\emptyset, \{\,\emptyset\,\}, \{\,\emptyset, \{\,\emptyset\,\}\,\}\,\}.
\end{aligned}
$$

Inductive Sets

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\mathrm{S}(\{\,\emptyset, \{\,\emptyset\,\}\,\}) &\doteq \{\,\emptyset, \{\,\emptyset\,\}\,\} \cup \{\,\{\,\emptyset, \{\,\emptyset\,\}\,\}\,\} && \doteq \{\,\emptyset, \{\,\emptyset\,\}, \{\,\emptyset, \{\,\emptyset\,\}\,\}\,\}.
\end{aligned}
$$

(b) Defining some natural numbers

$$1 \overset{\text{def}}{=} \mathrm{S}(0) \doteq 0 \cup \{\,0\,\} \doteq \{\,\emptyset\,\},$$

$$2 \overset{\text{def}}{=} \mathrm{S}(1) \doteq 1 \cup \{\,1\,\} \doteq \{\,\emptyset, \{\,\emptyset\,\}\,\},$$

$$3 \overset{\text{def}}{=} \mathrm{S}(2) \doteq 2 \cup \{\,2\,\} \doteq \{\,\emptyset, \{\,\emptyset\,\}, \{\,\emptyset, \{\,\emptyset\,\}\,\}\,\}.$$

Inductive Sets

Notation

$a + 1 \stackrel{\text{def}}{=} \mathrm{S}(a)$.

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Definition [3.]1.2

A set I is **inductive** if

(a) $\emptyset \in I$ and
(b) if $n \in I$ then $n + 1 \in I$.

Inductive Sets

Notation

$a + 1 \stackrel{\text{def}}{=} \mathrm{S}(a)$.

Definition [3.]1.2

A set I is **inductive** if

(a) $\emptyset \in I$ and

(b) if $n \in I$ then $n + 1 \in I$.

Remark

An inductive set will be an infinite set.

Outline

The Set of Natural Numbers

Definition [3.]1.3

The set of **all natural numbers**, denoted $\mathbf{N}$, is defined by

$$\mathbf{N} \overset{\text{def}}{=} \{\, x \mid x \in I \text{ for every inductive set } I \,\}.$$

The Set of Natural Numbers

Question

Is $\mathbf{N}$ a set?

The Set of Natural Numbers

Question

Is $\mathbf{N}$ a set?

Answer (partial)

Let A be an inductive set. Using the Axiom Scheme of Comprehension we can defined the set of all natural numbers by

$$\mathbf{N} \overset{\text{def}}{=} \{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

The Set of Natural Numbers

Question

Are there inductive sets?

The Set of Natural Numbers

Question

Are there inductive sets?

Remark

So far we only have the set $\emptyset$ and the axioms have the form: For every set X, there exists a set Y such that . . . .

The Set of Natural Numbers

The Axiom of Infinite

- *"An inductive set exits" (p. 41)*
- *"There exists an inductive set" (Enderton 1977)*
- $\exists A[\,\emptyset \in A \land \forall a(a \in A \to \mathrm{S}(a) \in A)\,].$

The Set of Natural Numbers

The Axiom of Infinite

- *"An inductive set exits" (p. 41)*
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Historical remark

Zermelo ([1908] 1967) did not state the Axiom of Infinite using the successor $\mathrm{S}(a)$ but the singleton set $\{\, a \,\}$.

The Set of Natural Numbers

"Some mathematicians object to the Axiom of Infinity on the grounds that a collection of objects produced by an infinite process (such as $\mathbf{N}$*) should not be treated as a completed entity. However, most people with some mathematical training have no difficulty visualizing the collection of natural numbers in that way. Infinite sets are basic tools of modern mathematics and the essence of set theory. No contradiction resulting from their use has ever been discovered in spite of the enormous body of research founded on them. Therefore, we treatthe Axiom of Infinity on a par with our other axioms." (p. 41)*

The Set of Natural Numbers

Lemma [3.]1.4

(a) The set $\mathbf{N}$ is inductive.

(b) If I is an inductive set, then $\mathbf{N} \subseteq I$.

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Remark

The set $\mathbf{N}$ is the smallest inductive set.

Ordering on Natural Numbers

An "extra" property of natural numbers

$$0 \in 1 \in 2 \in 3 \in \cdots$$

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Definition [3.]1.5

For all $m, n \in \mathbf{N}$, $m < n \stackrel{\text{def}}{=} m \in n$.

Ordering on Natural Numbers

An “extra” property of natural numbers

$$0 \in 1 \in 2 \in 3 \in \cdots$$

Definition [3.]1.5

For all $m, n \in \mathbf{N}$, $m < n \stackrel{\text{def}}{=} m \in n$.

Theorem

The pair $(\mathbf{N}, <)$ is a linearly ordered set.

Impredicative Definitions

A wrong impredicative definition

$$n \stackrel{\text{def}}{=} \{ 0, 1, \ldots, n-1 \}.$$

> *"We cannot just say that a set n is a natural number if its elements are all the smaller natural numbers, because such a 'definition' would involve the very concept being defined." (p. 40)*

Induction as Foundations

"Thus inductive definibility is a notion intermediate in strength between predicate and fully impredicative definability. It would be interesting to formulate a coherent conceptual framework that made induction the principal notion. There are suggestions of this in the literature, but the possibility has not yet been fully explored." (Aczel 1977, p. 780)

Should We Defining Natural Numbers as Sets?

Remark

So far, we defined natural numbers on terms of sets. A different point of view is stated by some authors (see, e.g. Benacerraf (1965)).

Zermelo's natural numbers

$$
\begin{aligned}
0 &\doteq \emptyset, \\
1 &\doteq \{\,0\,\} \doteq \{\,\emptyset\,\}, \\
2 &\doteq \{\,1\,\} \doteq \{\,\{\,\emptyset\,\}\,\}, \\
3 &\doteq \{\,2\,\} \doteq \{\,\{\,\{\,\emptyset\,\}\,\}\,\}, \\
&\vdots
\end{aligned}
$$

von Neumann's natural numbers

$$
\begin{aligned}
0 &\doteq \emptyset, \\
1 &\doteq \{\,0\,\} &&\doteq \{\,\emptyset\,\}, \\
2 &\doteq \{\,0, 1\,\} &&\doteq \{\,\emptyset, \{\,\emptyset\,\}\,\}, \\
3 &\doteq \{\,0, 1, 2\,\} &&\doteq \{\,\emptyset, \{\,\emptyset\,\}, \{\,\emptyset, \{\,\emptyset\,\}\,\}\,\}, \\
&\vdots
\end{aligned}
$$

Outline

References

Peter Aczel (1977). An Introduction to Inductive Definitions. In: Handbook of Mathematical Logic. Ed. by Jon Barwise. Vol. 90. Studies in Logic and the Foundations of Mathematics. Elsevier. Chap. C.7. DOI: 10.1016/S0049-237X(08)71120-0 (cit. on p. 30).

Paul Benacerraf (1965). What Numbers Could not Be. The Philosophical Review 74.1, pp. 47–73. DOI: 10.2307/2183530 (cit. on p. 31).

Herbert B. Enderton (1977). Elements of Set Theory. Academic Press (cit. on pp. 21, 22).

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 3).

Ernst Zermelo [1908] (1967). Investigations in the Foundations of Set Theory I. In: From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Ed. by Jean van Heijenoort. Source Books in the History of the Sciences. Harvard University Press, pp. 199–215 (cit. on pp. 21, 22).