CM0832 - MT5001 Elements of Set Theory
2.3 Functions

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Outline

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Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

Definition

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Example

Whiteboard.

Definition

Notation (p. 24)

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If F is a function with $\operatorname{dom} F$ we shall also use the following notations for the function F:

(a) $\langle F(a) \mid a \in A \rangle$,

(b) $\langle F_a \mid a \in A \rangle$ or

(c) $\langle F_a \rangle_{a \in A}$.

Definition

Discussion

Quote from the textbook:

> *"Function, as understood in mathematics, is a procedure, a rule, assigning to any object a from the domain of the function a unique object b, the value of the function at a." (p. 28)*

Is there any (implicit or explicit) procedure, rule or formula in our definition of function?

Definition

Remark

The current definition of a (one-value) function (of a real variable) is from Dirichlet who in 1837 wrote:

> *"If a variable y is so related to a variable x that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a **function** of the independent variable x."*
> *(Merzcbach and Boyer [1968] 2011, p. 452).*

Definition

Remark

The current definition of a function on arbitrary sets is from Cantor 1895:

> *"By a **covering of the aggregate** N **with elements of the aggregate**, M or, more simply, by a **covering of** N **with** M, we understand a law by which with every element n of N a definite element of M is bound up, where one and the same element of M can come repeatedly into application. The element of M bound up with n is, in a way, a one-valued function of n, and may be denoted by $f(n)$; it is called a **covering function of** n. The corresponding covering of N will be called $f(N)$." (Cantor [1895] 1915, p. 94)*

Functions Extensionality

Lemma [2.]3.2 (function extensionality)

Let F and G be functions. $F \doteq G$ if and only if $\operatorname{dom} F \doteq \operatorname{dom} G$ and $F(x) \doteq G(x)$, for all $x \in \operatorname{dom} F$.

Outline

One-To-One Functions

Definitions [2.]3.3

Let F be a function and A and B sets.

(a) F is a function **on** (**from**) A if and only if $\operatorname{dom} F \doteq A$.

(b) F is a function **into** (**to**) B if and only if $\operatorname{ran} F \subseteq B$.

(c) F is a function **onto** B if and only if $\operatorname{ran} F \doteq B$.

(d) The **restriction** of F to A, denoted $F \upharpoonright A$, is the function defined by

$$F \upharpoonright A \stackrel{\text{def}}{=} \{\, (x, y) \in F \mid x \in A \,\}.$$

Functions

Definition [2.]3.7

A function F is **one-to-one** (or **injective**) if for each $y \in \operatorname{ran} F$ there is only one x such that xFy. In other words, if $x_1, x_2 \in \operatorname{dom} F$ and $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

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Example

Whiteboard.

Outline

Compatible Systems of Functions

Definition [2.]3.10

(a) Functions f and g are **compatible** if $f(x) \doteq g(x)$ for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(b) A set of functions F is a **compatible system of functions** if any two functions f and g from F are compatible.

Outline

Sets of Functions

Definition [2.]3.13

Let A and B be sets. The **set of functions** on A into B, denoted B^A, is defined by

$$B^A \stackrel{\text{def}}{=} \{\, F \mid F : A \to B \,\}.$$

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Example

- $\{0, 1\}^{\mathbf{N}}$: The set of infinity binary sequences.

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Example

- $\{0, 1\}^{\mathbf{N}}$: The set of infinity binary sequences.
- $\emptyset^A \doteq \emptyset$ for $A \not\doteq \emptyset$ (no function can have a non-empty domain and an empty range).
- $A^\emptyset \doteq \{\emptyset\}$ for any set A ($\emptyset$ is the only function with an empty domain).

Sets of Functions

Question

Let A and B be sets. Is B^A a set?

Sets of Functions

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Let A and B be sets. Is B^A a set?

Answer

Definition of B^A without using the textbook conventions.

$$B^A \stackrel{\text{def}}{=} \{ F \in \mathcal{P}(A \times B) \mid F : A \to B \}.$$

That is, B^A is a set defined from the Axiom Scheme of Comprehension.

Outline

Products of Indexed Systems of Sets

Convention (p. 27)

Let $S = \langle S_i \mid i \in I \rangle$ be a function with domain I.

> *"We call the function $\langle S_i \mid i \in I \rangle$ an **indexed system of sets**, whenever we wish to stress that the values of S are sets."*

Products of Indexed Systems of Sets

Definition (p. 27)

Let $S = \langle S_i \mid i \in I \rangle$ be an indexed system of sets. The **product** (or **generalised product**) of S, denoted $\prod S$, is defined by

$$\prod S \stackrel{\text{def}}{=} \{\, f \mid f \text{ is a function on } I \text{ and } f_i \in S_i, \text{for all } i \in I \,\}$$
$$\doteq \left\{\, f \,\middle|\, f : I \to \bigcup \{\, S_i \mid i \in I \,\} \text{ and } f_i \in S_i, \text{for all } i \in I \,\right\}.$$

Products of Indexed Systems of Sets

Example (informal)

Let A and B be two sets, $I = \{0, 1\}$ and let $S = \langle S_i \mid i \in I \rangle$ be an indexed system of sets where $S_0 = A$ and $S_1 = B$. Then

$$\prod S \doteq \left\{ f \;\middle|\; f : I \to \bigcup\{A, B\}, \text{ such that } f_0 \in A \text{ and } f_1 \in B \right\}.$$

Now, we can define the one-to-one correspondence

$$h : \prod S \to A \times B$$
$$h(f) = (f_0, f_1).$$

Products of Indexed Systems of Sets

Remark

Let $\langle S_i \mid i \in I \rangle$ be an indexed system of sets. If $S_i = B$ for all $i \in I$, then

$$\begin{aligned} \prod S &\doteq B^I \\ &\doteq \{\, f \mid f : I \to B \,\}. \end{aligned}$$

Products of Indexed Systems of Sets

Notation (p. 27)

Let $\prod S$ be the product of an indexed system of sets $S = \langle S_i \mid i \in I \rangle$. We shall also use the following notations for the product $\prod S$:

(a) $\prod \langle S_i \mid i \in I \rangle$,

(b) $\prod_{i \in I} S(i)$ or

(c) $\prod_{i \in I} S_i$.

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Let $\prod S$ be the product of an indexed system of sets $S = \langle S_i \mid i \in I \rangle$.
Is $\prod S$ a set?

Answer

Definition of $\prod S$ without using the textbook conventions.

$$\prod S \stackrel{\text{def}}{=} \left\{ f \in \mathcal{P}\left(I \times \bigcup \{ S_i \mid i \in I \}\right) \,\middle|\, f \text{ is a function on } I \text{ and } f_i \in S_i, \text{for all } i \in I \right\}$$

That is, $\prod S$ is a set defined from the Axiom Scheme of Comprehension.

Outline

References

Georg Cantor (1895). Beiträge zur Begründung der transfiniten Mengenlehre. Mathematische Annalen 46.4, pp. 481–512. DOI: 10.1007/BF02124929 (cit. on p. 38).

— [1895] (1915). Contributions to the Founding of the Theory of Transfinite Numbers. Translation of Cantor (1895). Translated, and provided with an introduction and notes, by Philip E. B. Jourdain. Dover Publications (cit. on p. 14).

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 4).

Uta C. Merzcbach and Carl B. Boyer [1968] (2011). A History of Mathematics. 3rd ed. John Wiley & Sons (cit. on p. 13).