CM0832 - MT5001 Elements of Set Theory
1.3 The Axioms

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Outline

Preliminaries

Textbook

Karel Hrbacek and Thomas Jech ([1978] 1999). Introduction to Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Outline

The Axiom of Existence

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- *"There exists a set which has no elements." (p. 7)*
- $(\exists B)(\forall x)(x \notin B).$

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The Axiom of Existence postulate the our universe of discourse is not void. This axiom postulates the existence of the empty set.

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The Axiom of Existence postulate the our universe of discourse is not void. This axiom postulates the existence of the empty set.[†]

[†]Strictly speaking, the universe of the discourse of FOL is not empty. For example, let $\mathbf{P}(x)$ be a property, then $(\forall x)\mathbf{P}(x) \rightarrow (\exists x)\mathbf{P}(x)$ is a theorem. See, e.g. (Lambert 2001).

Outline

The Axiom of Extensionality

The Axiom of Extensionality

- *"If every element of X is an element of Y and every element of Y is an element of X, then $X \doteq Y$." (p. 7)*
- *"If two sets have exactly the same members, then they are equal." (Enderton 1977)*
- $(\forall X)(\forall Y)[(\forall z)(z \in X \leftrightarrow z \in Y) \to X \doteq Y].$

The Axiom of Extensionality

Lemma [1.]3.1

There exists only one set with no elements.

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Proof

On the whiteboard.

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Definition [1.]3.2

The **empty set**, denoted $\emptyset$, is set with no elements.

The Axiom of Extensionality

Theorem (converse of the Axiom of Extensionality)

If $X \doteq Y$, then every element of X is an element of Y and every element of Y is an element of X.

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$$(\forall X)(\forall Y)[\, X \doteq Y \to (\forall z)(z \in X \leftrightarrow z \in Y)\,].$$

Proof

FOL.

The Axiom of Extensionality

Theorem (converse of the Axiom of Extensionality)

If $X \doteq Y$, then every element of X is an element of Y and every element of Y is an element of X.[†]

$$(\forall X)(\forall Y)[\, X \doteq Y \to (\forall z)(z \in X \leftrightarrow z \in Y)\,].$$

Proof

FOL.

[†]Due to this theorem some authors (see, e.g. (Goldrei 1996, p. 76)) state the Axiom of Extensionality using a bi-conditional, that is, if every element of X is an element of Y and every element of Y is an element of X if and only if $X \doteq Y$. $(\forall X)(\forall Y)[\,(\forall z)(z \in X \leftrightarrow z \in Y) \leftrightarrow X \doteq Y\,]$.

The Axiom of Extensionality

Theorem

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(b) if $X \doteq Y$ and $Z \in X$, then $Z \in Y$.

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(b) if $X \doteq Y$ and $Z \in X$, then $Z \in Y$.

Proof

(a) Hint. To use that the identity is substitutive and the property $\mathbf{P}(A, B)$: "$A \in B$".

(b) Hint. To use the converse of the Axiom of Extensionality.

Outline

The Axiom Schema of Comprehension

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- *"Let* $\mathbf{P}(x)$ *be a unary property of* x*. For any set* A*, there is a set* B *such that* $x \in B$ *if and only if* $x \in A$ *and* $\mathbf{P}(x)$*." (p. 8)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

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- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \land \mathbf{P}(x))$.

Remark

We postulate an axiom schema, i.e., there is an axiom for each property $\mathbf{P}(x)$.

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- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Example

Let $\mathbf{P}(x)$: "$x \notin x$". The axiom postulates:

For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $x \notin x$ (the set B is the empty set).

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- *"Let $\mathbf{P}(x)$ be a unary property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$." (p. 8)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Remark

> *"Although the supply of axioms is unlimited, this causes no problems, since it is easy to recognize whether a particular statement is or is not an axiom and since every proof uses only finitely many axioms." (p. 8)*

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- *"Let $\mathbf{P}(x)$ be a unary property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$." (p. 8)*
- $(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x))$.

Remark

When using the axiom scheme with a $(n+1)$-ary property $\mathbf{P}(x, p_1, \ldots, p_n)$ the axiom is:

For any sets $p_1, \ldots, p_n, A$, there is a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x, p_1, \ldots, p_n)$.

$$(\forall p_1) \cdots (\forall p_n)(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x, p_1, \ldots, p_n)).$$

The Axiom Schema of Comprehension

Example [1.]3.3

To prove that if A and B are sets, then there is a set C such that $x \in C$ if and only if $x \in A$ and $x \in B$.

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Proof

Hint: To define the binary property $\mathbf{P}(x, B)$: "$x \in B$" and use the Axiom Schema of Comprehension.

The Axiom Schema of Comprehension

Question

Let $\mathbf{P}(x)$: "$x \notin x$". Without using the Axiom of Existence can we prove the existence of the empty set from the Axiom Schema of Comprehension and the property $\mathbf{P}(x)$?

The Axiom Schema of Comprehension

Lemma [1.]3.4

For every A, there is only one set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

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For every A, there is only one set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Remark

This lemma establishes that the set postulated by the Axiom Schema of Comprehension is unique.

The Axiom Schema of Comprehension

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For every A, there is only one set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Proof

On the whiteboard.

The Axiom Schema of Comprehension

Lemma [1.]3.4

For every A, there is only one set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Question

Given the above lemma, why not state the Axiom Schema of Comprehension by "for any set A, there is a unique set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$"?

$$(\forall A)(\exists! B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \mathbf{P}(x)).$$

The Axiom Schema of Comprehension

Definition [1.]3.5

Given a set A and a property $\mathbf{P}(x)$, the set B postulated by the Axiom Schema of Comprehension is

$$\{ x \in A \mid \mathbf{P}(x) \}.$$

The Axiom Schema of Comprehension

Definition [1.]3.5

Given a set A and a property $\mathbf{P}(x)$, the set B postulated by the Axiom Schema of Comprehension is

$$\{\, x \in A \mid \mathbf{P}(x) \,\}.^{\dagger}$$

† Strictly speaking before this definition we should introduce the abstraction symbol $\{\, \cdot \mid \cdot \,\}$ in the language of ZFC. See, e.g. (Drake 1974).

Outline

The Axiom of Pair

- *"For any A and B, there is a set C such that $x \in C$ if and only if $x \doteq A$ or $x \doteq B$." (p. 9)*
- *"For any sets A and B, there is a set having as members just A and B." (Enderton 1977)*
- $(\forall A)(\forall B)(\exists C)(\forall x)(x \in C \leftrightarrow x \doteq A \vee x \doteq B)$.

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- *"For any sets A and B, there is a set having as members just A and B." (Enderton 1977)*
- $(\forall A)(\forall B)(\exists C)(\forall x)(x \in C \leftrightarrow x \doteq A \lor x \doteq B)$.

Exercise

To prove that the set postulated by the axiom is unique.

The Axiom of Pair

Definition (p. 9)

The **unordered pair** of A and B, denoted $\{A, B\}$, is the set postulated by the Axiom of Pair.

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$\{A\} \overset{\text{def}}{=} \{A, A\}$.

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$\{A\} \stackrel{\text{def}}{=} \{A, A\}$.

Definition (p. 17)

Let A be a set. The **singleton** of A, denoted $\{A\}$, is the set that has A as its only element.

Outline

The Axiom of Union

The Axiom of Union

- *"For any set S, there exists a set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$." (p. 9)*
- *"For any set S, there exists a set U whose elements are exactly the members of the members of A". (Enderton 1977)*
- $(\forall S)(\exists U)(\forall x)[\, x \in U \leftrightarrow (\exists A)(x \in A \wedge A \in S)\,].$

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- *"For any set S, there exists a set U whose elements are exactly the members of the members of A". (Enderton 1977)*
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Definition (p. 9)

The (**generalised**) **union** of S, denoted $\bigcup S$, is the set postulated by the axiom.

The Axiom of Union

Examples

- Let A, B sets. Then, $x \in \bigcup\{A, B\}$ if and only if $x \in A$ or $x \in B$.

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- Let A, B sets. Then, $x \in \bigcup\{A, B\}$ if and only if $x \in A$ or $x \in B$.
- $\bigcup\emptyset \doteq \emptyset$.

Definition (p. 10)

The **union** of A and B, denoted $A \cup B$, is the set $\bigcup\{A, B\}$.

The Axiom of Union

Definition [1.]3.10

A set A is a **subset** of a set B, denoted $A \subseteq B$, if every element of A belongs to B.

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A set A is a **subset** of a set B, denoted $A \subseteq B$, if every element of A belongs to B.

Example [1.]3.11

(a) $\emptyset \subseteq A$ and $A \subseteq A$ for every set A.

(b) $\{\, x \in A \mid \mathbf{P}(x) \,\} \subseteq A$.

(c) If $A \in S$, then $A \subseteq \bigcup S$.

Outline

The Axiom of Power Set

- *"For any set S, there exists a set P such that $X \in P$ if and only if $X \subseteq S$." (p. 10)*
- *"For any set S, there is a set whose members are exactly the subsets of S." (Enderton 1977)*
- $(\forall S)(\exists P)(\forall X)[X \in P \leftrightarrow X \subseteq S]$.

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Exercise

To prove that the set postulated by the axiom is unique.

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- $(\forall S)(\exists P)(\forall X)[\, X \in P \leftrightarrow X \subseteq S \,].$

Definition (p. 10)

The **power set** of S, denoted $\mathcal{P}(S)$, is the set postulated by the axiom, i.e., is the set of all subsets of S.

The Axiom of Power Set

Convention (p. 11)

If it has been proved that some set A contains all x with the property $\mathbf{P}(x)$ then

$$\{\, x \mid \mathbf{P}(x) \,\} \stackrel{\text{def}}{=} \{\, x \in A \mid \mathbf{P}(x) \,\}.$$

Outline

References

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