CM0832 Elements of Set Theory
3. Relations and Functions

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Preliminaries

Textbook

Enderton (1977). Elements of Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Ordered Pairs

Observation

Let a and b be sets. An ordered pair $\langle a, b\rangle$ should be a set such that

$$\langle a, b\rangle = \langle c, d\rangle \quad \text{iff} \quad a = c \wedge b = d.$$

Definition

We define an **ordered pair** using Kuratowski's definition, that is,

$$\langle a, b\rangle := \{\{a\}, \{a, b\}\}.$$

Ordered Pairs

Example

We show that $\langle \emptyset, \{\emptyset\} \rangle \neq \langle \{\emptyset\}, \emptyset \rangle$.

$$\begin{aligned} \langle \emptyset, \{\emptyset\} \rangle &= \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &= \{\{\emptyset\}, \{\{\emptyset\}, \emptyset\}\} \\ &\neq \{\{\{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\} \\ &= \langle \{\emptyset\}, \emptyset \rangle. \end{aligned}$$

Ordered Pairs

Example

Let a be a set. Then

$$\begin{aligned}\langle a, a\rangle &= \{\{a\}, \{a, a\}\}\\ &= \{\{a\}, \{a\}\}\\ &= \{\{a\}\}.\end{aligned}$$

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Exercise

To give a different definition of ordered pair.

Cartesian Product

Definition

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$$A \times B := \{\, \langle x, y \rangle \mid x \in A \wedge y \in B \,\}.$$

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Let A and B be sets. Note that $A \times B$ is a set because we can define it via the subset axiom scheme.

$$A \times B := \{ \langle x, y \rangle \in \mathcal{P}\mathcal{P}(A \cup B) \mid x \in A \wedge y \in B \}.$$

Relations

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A **relation** is a set of ordered pairs.

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Let R the relation defined by $R = \{\langle a, b\rangle, \langle b, b\rangle, \langle c, b\rangle\}$. Diagram: whiteboard.

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Let R the relation defined by $R = \{\langle a, b\rangle, \langle b, b\rangle, \langle c, b\rangle\}$. Diagram: whiteboard.

Example

Let $\omega = \{0, 1, 2, \dots\}$. The identity relation on ω is defined by

$$\begin{aligned} \mathrm{I}_\omega &:= \{\, \langle n, n\rangle \mid n \in \omega \,\} \\ &= \{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \dots\}. \end{aligned}$$

Relations

Definition

Let R be a relation. We define the **domain**, the **range** and the **field** of R by

$$\begin{aligned}
\operatorname{dom} R &:= \{\, x \mid \exists y(\langle x, y\rangle \in R) \,\},\\
\operatorname{ran} R &:= \{\, y \mid \exists x(\langle x, y\rangle \in R) \,\},\\
\operatorname{fld} R &:= \operatorname{dom} R \cup \operatorname{ran} R.
\end{aligned}$$

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Observation

Let R be a relation. Note that $\operatorname{dom} R$ and $\operatorname{ran} R$ are sets because we can define them via the subset axiom scheme.

$$\begin{aligned} \operatorname{dom} R &:= \left\{\, x \in \bigcup\bigcup R \;\middle|\; \exists y(\langle x, y\rangle \in R) \,\right\}, \\ \operatorname{ran} R &:= \left\{\, y \in \bigcup\bigcup R \;\middle|\; \exists x(\langle x, y\rangle \in R) \,\right\}. \end{aligned}$$

n-Ary Relations

Definition

We define an **ordered n-tuple**, for $n \geq 3$, by

$$\langle x_1, x_2, \ldots, x_n \rangle := \langle \langle x_1, x_2, \ldots, x_{n-1} \rangle, x_n \rangle$$

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Example

Ordered triple (3-tuple) and ordered quadruple (4-tuple).

$$\langle x_1, x_2, x_3 \rangle := \langle \langle x_1, x_2 \rangle, x_3 \rangle,$$
$$\langle x_1, x_2, x_3, x_4 \rangle := \langle \langle x_1, x_2, x_3 \rangle, x_4 \rangle.$$

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$$\langle x_1, x_2, x_3, x_4 \rangle := \langle \langle x_1, x_2, x_3 \rangle, x_4 \rangle.$$

Definition

We define an 1-**tuple** by

$$\langle x \rangle := x.$$

n-Ary Relations

Definition

Let A be a set. We define an n-**ary relation on** A to be a set of ordered n-tuples with all components in A.

n-Ary Relations

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Example

Whiteboard.

Observation

Let A be a set. Note that an 1-ary relation on A is just a subset of A but it is not a relation.

Functions

Definition

A **function** (**mapping** or **correspondence**) is a relation F such that for each x in $\operatorname{dom} F$ there is only one y such that xFy.

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Notation

We write $F : A \to B$ iff F is a function, $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.

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Notation

We write $F : A \to B$ iff F is a function, $\operatorname{dom} F = A$ and $\operatorname{ran} F \subseteq B$.

Definition

Let F be a function and A and B sets.

(i) F is a function **on** (**from**) A iff $\operatorname{dom} F = A$.

(ii) F is a function **into** (**to**) B iff $\operatorname{ran} F \subseteq B$.

(iii) F is a function **onto** B iff $\operatorname{ran} F = B$.

Functions

Exercise 3.11

Prove the following version (for functions) of the extensionality principle: Assume that F and G are functions, $\operatorname{dom} F = \operatorname{dom} G$, and $F(x) = G(x)$ for all x in the common domain. Then $F = G$.

Functions

Definition

A function F is **one-to-one** (or **injective**) iff for each $y \in \operatorname{ran} F$ there is only one x such that xFy. In other words, if $x_1, x_2 \in \operatorname{dom} F$ and $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

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A function F is an **one-to-one correspondence** between A and B iff F is an one-to-one function from A onto B.

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Functions

Definition

Let A, F and G be sets. We define, the **inverse** of F, the **composition** of F and G, the **restriction** of F to A and the **image** of A under F by

$$F^{-1} := \{\, \langle y, x\rangle \mid xFy \,\} \qquad \text{(inverse of } F\text{)}$$

$$F \circ G := \{\, \langle x, y\rangle \mid \exists t\,(xGt \wedge tFy) \,\} \qquad \text{(composition of } F \text{ and } G\text{)}$$

$$F \restriction A := \{\, \langle x, y\rangle \mid x \in A \wedge xFy \,\} \qquad \text{(restriction of } F \text{ to } A\text{)}$$

$$\begin{aligned} F[\![A]\!] &:= \operatorname{ran}(F \restriction A) \qquad \text{(image of } A \text{ under } F\text{)} \\ &= \{\, y \mid \exists x\,(x \in A \wedge xFy) \,\} \end{aligned}$$

Functions

Example

Let

$$F = \{\langle\emptyset, a\rangle, \langle\{\emptyset\}, b\rangle\}.$$

Then

$$\begin{aligned}
\operatorname{dom} F &= \{\emptyset, \{\emptyset\}\} \\
\operatorname{ran} F &= \{a, b\}, \\
F^{-1} &= \{\langle a, \emptyset\rangle, \langle b, \{\emptyset\}\rangle\}, \\
F \restriction \emptyset &= \emptyset, \\
F \restriction \{\emptyset\} &= \{\langle\emptyset, a\rangle\}, \\
F[\![\{\emptyset\}]\!] &= \{a\}, \\
F(\{\emptyset\}) &= b.
\end{aligned}$$

F is a function,

F^{-1} is function iff $a \neq b$,

Functions

Exercise 3.18

Let R be the set

$$\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}.$$

To find $R \circ R$, $R \restriction \{1\}$, $R^{-1} \restriction \{1\}$, $R[\![\{1\}]\!]$ and $R^{-1}[\![\{1\}]\!]$.

Functions

Exercise 3.18

Let R be the set

$$\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}.$$

To find $R \circ R$, $R \restriction \{1\}$, $R^{-1} \restriction \{1\}$, $R[\![\{1\}]\!]$ and $R^{-1}[\![\{1\}]\!]$.

Exercise (p. 44)

Let A, F and G be sets. Show that F^{-1}, $F \circ G$, $F \restriction A$ and $F[\![A]\!]$ are sets.

Functions

Theorem 3E

Let F be a set. Then

$$\operatorname{dom} F^{-1} = \operatorname{ran} F \quad \text{and} \quad \operatorname{ran} F^{-1} = \operatorname{dom} F.$$

If additionally F is a relation, then

$$(F^{-1})^{-1} = F.$$

Functions

Theorem 3G

Let F be an one-to-one function.

- If $x \in \operatorname{dom} F$, then
$$F^{-1}(F(x)) = x.$$
- If $y \in \operatorname{ran} F$, then
$$F(F^{-1}(y)) = y.$$

Functions

Theorem 3H

Let F and G be functions. Then

- $F \circ G$ is a function,
- $\operatorname{dom}(F \circ G) = \{\, x \in \operatorname{dom} G \mid G(x) \in \operatorname{dom} F \,\}$ and
- if $x \in \operatorname{dom}(F \circ G)$, then $(F \circ G)(x) = F(G(x))$.

Functions

Theorem 3I

Let F and G be sets. Then

$$(F \circ G)^{-1} = G^{-1} \circ F^{-1}.$$

Functions

Theorem 3J

Let F be a function $F : A \to B$ and $A \neq \emptyset$.

(i) There exists a function $G : B \to A$ (a "left inverse") such that $G \circ F$ is the identity function I_A on A iff the function F is one-to-one.

(ii) There exists a function $H : B \to A$ (a "right inverse") such that $F \circ H$ is the identity function I_B on B iff the function F maps A onto B.

Functions

Axiom of choice (first form)

For any relation R there is a function $H \subseteq R$ with $\operatorname{dom} H = \operatorname{dom} R$.

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Example

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Observation

Is the axiom of choice accepted in constructive mathematics? (See, e.g. Martin-Löf (2006)).

Functions

Definition

Let A and B be sets. We define the **set of functions** from A into B by

$$B^A := \{\, F \mid F : A \to B \,\} =: {}^A B.$$

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Example

- $\{0, 1\}^\omega$: The set of infinity binary sequences.

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- $\emptyset^A = \emptyset$ for $A \neq \emptyset$ (no function can have a non-empty domain and an empty range).

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Example

- $\{0, 1\}^\omega$: The set of infinity binary sequences.
- $\emptyset^A = \emptyset$ for $A \neq \emptyset$ (no function can have a non-empty domain and an empty range).
- $A^\emptyset = \{\emptyset\}$ for any set A ($\emptyset$ is the only function with an empty domain).

Functions

Observation

Let A and B be sets. Note that B^A is a set because we can define it via the subset axiom scheme.

$$B^A := \{ F \in \mathcal{P}(A \times B) \mid F : A \to B \}.$$

Families

Observation

Families is another way to express functions when the range of a function is more important than the function itself. We write functions as families when we want to put the emphasis on the values of the function rather in the function.[†]

[†] Enderton (1977) do not use families, but 'only' functions.

Families

Observation

Families is another way to express functions when the range of a function is more important than the function itself. We write functions as families when we want to put the emphasis on the values of the function rather in the function.†

Observation

The terminology and notation on families is not established.

† Enderton (1977) do not use families, but 'only' functions.

Families

Definition

Let I and X be sets. A **family in X indexed by I** is a function

$$A : I \to X$$
$$A = \{ \langle i, A_i \rangle \mid i \in I \text{ and } A_i \in X \},$$

where $A_i := A(i)$, for all $i \in I$.[†] The set I is the **index set** of the family.

[†]See, e.g. Halmos (1960), Drake (1974) and Hamilton ([1982] 1992).

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Notation

The above family A is denoted by $\langle A_i \mid i \in I \rangle$ following to (Hrbacek and Jech [1978] 1999).

[†]See, e.g. Halmos (1960), Drake (1974) and Hamilton ([1982] 1992).

Families

Definition

The **union** of a family $\langle A_i \mid i \in I \rangle$ is defined by

$$\begin{aligned}\bigcup_{i \in I} A_i &:= \bigcup \{ A_i \mid i \in I \} \\ &= \{ x \mid x \in A_i \text{ for some } i \text{ in } I \}.\end{aligned}$$

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Example

Whiteboard.

Families

Definition

The **intersection** of a family $\langle A_i \mid i \in I \rangle$ is defined by

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Example

Whiteboard.

Families

Definition

The **Cartesian product** (or **generalised product**) of a family $\langle A_i \mid i \in I \rangle$ is defined by

$$\bigtimes_{i \in I} A_i := \left\{ f \;\middle|\; f : I \to \bigcup_{i \in I} A_i \text{ and } \forall i \, (i \in I \to f(i) \in A_i) \right\} =: \prod_{i \in I} A_i.$$

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Example

Let $\langle A_i \mid i \in I \rangle$ be a family. If $A_i = B$ for all $i \in I$, then

$$\begin{aligned} \bigtimes_{i \in I} A_i &= B^I \\ &= \{ f \mid f : I \to B \}. \end{aligned}$$

Families

Example

The following example illustrates the generalisation of the Cartesian product.

Let X and Y be two sets. Recall that the Cartesian product of X and Y was defined by

$$X \times Y := \{ \langle x, y \rangle \mid x \in X \wedge y \in Y \}.$$

(continued on next slide)

Families

Example (continuation)

Let $I = \{a, b\}$ be an index set and let $\langle Z_i \mid i \in I \rangle$ be a family where $Z_a = X$ and $Z_b = Y$. Then

$$\bigtimes_{i \in I} Z_i = \{\, f \mid f : I \to X \cup Y, \text{ such that } f(a) \in X \text{ and } f(b) \in Y \,\}.$$

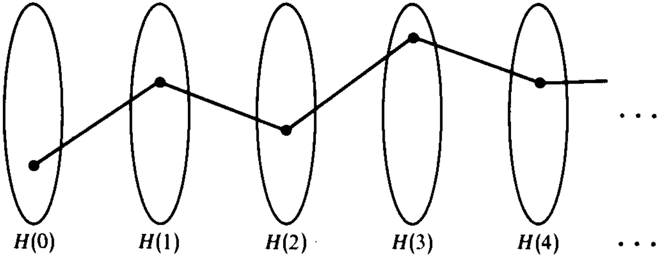
Now, we can define the one-to-one correspondence

$$h : \bigtimes_{i \in I} Z_i \to X \times Y$$
$$h(f) = \langle f(a), f(b) \rangle.$$

Families

Axiom of choice (second form)

Let $\langle H_i \mid i \in I \rangle$ be a family. If $H(i) \neq \emptyset$ for all $i \in I$, then $\bigtimes_{i \in I} H(i) \neq \emptyset$.[†]



[†]Figure source: Enderton (1977, Fig. 11)

Equivalence Relations

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Let R be a binary relation on a set A. The relation R is

- **reflexive** iff xRx for all $x \in A$,

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- **reflexive** iff xRx for all $x \in A$,
- **symmetric** iff xRy implies yRx for all $x, y \in A$ and
- **transitive** iff xRy and yRz imply xRz for all $x, y, z \in A$.

Example

Whiteboard.

Equivalence Relations

Introduction

Whiteboard.

Equivalence Relations

Introduction

Whiteboard.

Definition

Let R be a binary relation on a set A. The relation R is an **equivalence relation** iff R is reflexive, symmetric and transitive.

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Equivalence Relations

Questions

- Let $A = \{a, e, i, o, u\}$. Is the equality relation on A an equivalence relation?

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Equivalence Relations

Questions

- Let $A = \{a, e, i, o, u\}$. Is the equality relation on A an equivalence relation?
- Let $A \neq \emptyset$ be a set. Is the relation $\emptyset$ on A an equivalence relation?
- Let A be a set. Is the relation $A \times A$ an equivalence relations?
- Let A be a singleton. It is possible to define an equivalence relation on A?

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Definition

Let R be an equivalence relation on a set A and let $x \in \operatorname{fld} R$. The set $[x]_R$ is the **equivalence class of x (modulo R)**.

Notation

We write $[x]$ if the relation R is clear in the context.

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Example

Whiteboard.

Equivalence Relations

Theorem 3N

Let R be an equivalence relation on a set A and let $x, y \in A$. Then

$$[x]_R = [y]_R \quad \text{iff} \quad xRy.$$

Equivalence Relations

Theorem 3P

Let R be an equivalence relation on a set A. Then the set

$$\{ [x]_R \mid x \in A \}$$

of all equivalence classes is a partition of the set A.

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of all equivalence classes is a partition of the set A.

Exercise 3.37

Assume that Π is a partition of a set A. Define the relation R_Π as follows:

$$x R_\Pi y \quad \text{iff} \quad (\exists B \in \Pi)(x \in B \wedge y \in B).$$

Show that R_Π is an equivalence relation on A.

Equivalence Relations

Definition

Let R be an equivalence relation on a set A. The **quotient set** is defined by

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$$f(x) = [x]_R.$$

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Observation

Using the λ-notation we could define the natural map by the anonymous function $\lambda x.[x]_R$.

Linear Ordering Relations

Motivation

What means that R is an ordering relation on a set A?

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What means that R is an ordering relation on a set A?

Definition

Let R be a binary relation on a set A. The relation R satisfies **trichotomy** if exactly one of the three alternatives

$$xRy, \quad x = y \quad \text{or} \quad yRx$$

holds for all $x, y \in A$.

Linear Ordering Relations

Definition

Let A be a set. A **linear ordering** (or **total ordering**) on A is a binary relation R on A such that:

(i) R is transitive relation and

(ii) R satisfies trichotomy.

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Let A be a set. A **linear ordering** (or **total ordering**) on A is a binary relation R on A such that:

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(ii) R satisfies trichotomy.

Example

References

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