CM0832 Elements of Set Theory
7. Orderings and Ordinals

Andrés Sicard-Ramírez

Universidad EAFIT

Semester 2017-2

Preliminaries

Textbook

Enderton (1977). Elements of Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Well-Orderings

Definition

A **well-ordering** on A is a linear ordering on A with the further property that every non-empty subset of A has a least element.

Well-Orderings

Definition

A **well-ordering** on A is a linear ordering on A with the further property that every nonempty subset of A has a least element.

Definition

A **structure** is a pair $\langle A, R\rangle$ consisting of a set A and a binary relation R on A.

Transfinite Induction Principle

Definition

Let $<$ be some sort of ordering on A and $t \in A$. The **initial segment up to t** is the set

$$\operatorname{seg} t := \{\, x \in A \mid x < t \,\}.$$

Transfinite Induction Principle

Definition

Let $<$ be some sort of ordering on A and $t \in A$. The **initial segment up to t** is the set

$$\operatorname{seg} t := \{\, x \in A \mid x < t \,\}.$$

Transfinite induction principle

Let $\langle A, < \rangle$ be a well-ordered structure and assume that B is a subset of A with the special property that for every t in A,

$$\operatorname{seg} t \subseteq B \quad \text{implies} \quad t \in B.$$

Then B coincides with A.

Transfinite Recursion Theorem

Definition

Let $\langle A, < \rangle$ be a well-ordered structure and let B a set. The set of **all functions from initial segments of $\langle A, < \rangle$ into B** is defined by

$$B^{A<} := \{ f \mid f : \operatorname{seg} t \to B, \text{ for some } t \in A \}.$$

Transfinite Recursion Theorem

Definition

Let $\langle A, <\rangle$ be a well-ordered structure and let B a set. The set of **all functions from initial segments of $\langle A, <\rangle$ into B** is defined by

$$B^{A<} := \{\, f \mid f : \operatorname{seg} t \to B, \text{ for some } t \in A \,\}.$$

Observation

Let $\langle A, <\rangle$ be a well-ordered structure and let B a set. Note that $B^{A<}$ is a set because we can define it via the subset axiom scheme.

$$B^{A<} := \{\, f \in \mathcal{P}(A \times B) \mid f : \operatorname{seg} t \to B, \text{ for some } t \in A \,\}.$$

Transfinite Recursion Theorem

Transfinite recursion theorem (preliminary form, p. 175)

Let $\langle A, < \rangle$ be a well-ordered structure and let $G : B^{A<} \to B$. Then there is a unique function F such that for any $t \in A$,

$$F : A \to B$$
$$F(t) = G(F \restriction \operatorname{seg} t).$$

Replacement Axiom Scheme

Replacement axiom scheme

For any propositional function $\varphi(x, y)$, not containing B, the following is an axiom:

$$\forall A \, [\, \forall x \, (x \in A \to \exists! y \, \varphi(x, y)) \to \exists B \, \forall y \, (y \in B \leftrightarrow \exists x \, (x \in A \land \varphi(x, y))) \,].$$

Replacement Axiom Scheme

Replacement axiom scheme

For any propositional function $\varphi(x, y)$, not containing B, the following is an axiom:

$$\forall A \, [\, \forall x \, (x \in A \to \exists! y \, \varphi(x, y)) \to \exists B \, \forall y \, (y \in B \leftrightarrow \exists x \, (x \in A \wedge \varphi(x, y))) \,].$$

Observation

We stated an axiom scheme.

Replacement Axiom Scheme

Replacement axiom scheme

For any propositional function $\varphi(x, y)$, not containing B, the following is an axiom:

$$\forall A \, [\, \forall x \, (x \in A \to \exists! y \, \varphi(x, y)) \to \exists B \, \forall y \, (y \in B \leftrightarrow \exists x \, (x \in A \wedge \varphi(x, y))) \,].$$

Observation

We stated an axiom scheme.

Abstraction from the replacement axiom scheme

$\{ \, y \mid \exists x \, (x \in A \wedge \varphi(x, y) \, \}$ is a set.

Replacement Axiom Scheme

Replacement axiom scheme

For any propositional function $\varphi(x, y)$, not containing B, the following is an axiom:

$$\forall A\,[\,\forall x\,(x \in A \to \exists! y\,\varphi(x, y)) \to \exists B\,\forall y\,(y \in B \leftrightarrow \exists x\,(x \in A \land \varphi(x, y)))\,].$$

Observation

We stated an axiom scheme.

Abstraction from the replacement axiom scheme

$\{\, y \mid \exists x\,(x \in A \land \varphi(x, y)\,\}$ is a set.

Observation

The propositional function φ can depend on other variables $t_1, \dots, t_k$. In this case, we use $\varphi(x, y, t_1, \dots, t_k)$ and we universally quantify on variables $t_1, \dots, t_k$ when using the axiom scheme.

Epsilon-Images[†]

[†] 'The membership symbol ($\in$) is not typographically the letter epsilon but originally it was, and the name "epsilon" persists.' (Enderton 1977, p. 182)

Isomorphisms

Definition

Let $\langle A, R\rangle$ and $\langle B, S\rangle$ be two structures. An **isomorphism** from $\langle A, R\rangle$ onto $\langle B, S\rangle$ is a one-to-one function f from A onto B such that for all $x, y \in A$

$$x \; R \; y \quad \text{iff} \quad f(x) \; S \; f(y).$$

Isomorphisms

Corollary 7H

Let α be the $\in$-image of a well-ordered structure $\langle A, < \rangle$. Then α is a transitive set and $\in_\alpha$ is a well ordering on α, where

$$\in_\alpha := \{ \langle x, y \rangle \in \alpha \times \alpha \mid x \in y \}.$$

Ordinal Numbers

Idea

To assign a 'number' to each well-ordered structure that measures its 'length'. Two well-ordered structures should receive the same number, if and only if, they are isomorphic.

Ordinal Numbers

Idea

To assign a ‘number’ to each well-ordered structure that measures its ‘length’. Two well-ordered structures should receive the same number, if and only if, they are isomorphic.

Theorem 7I

Two well-ordered structures are isomorphic iff they have the same $\in$-image.

Ordinal Numbers

Definition

Let $<$ be a well-ordering on A. The **ordinal number** of $\langle A, < \rangle$ is its ϵ-image. An **ordinal number** is a set that is the ordinal number of some well-ordered structure.

Ordinal Numbers

Definition

A set A is **well-ordered by the membership relation** iff the relation

$$\in_A := \{ \langle x, y \rangle \in A \times A \mid x \in y \}$$

is a well-ordering on A.

Ordinal Numbers

Definition

A set A is **well-ordered by the membership relation** iff the relation

$$\in_A := \{ \langle x, y \rangle \in A \times A \mid x \in y \}$$

is a well-ordering on A.

Definition (other definition of ordinal number)

A set A is an **ordinal number** iff (Hrbacek and Jech [1978] 1999, p. 107):

(i) The set is transitive.

(ii) The set is well-ordered by the membership relation.

Ordinal Numbers

Burali-Forti theorem (p. 194)

There is no set to which every ordinal number belongs.

Well-Ordering Theorem

Well-ordering theorem (p. 196)

For any set A, there is a well-ordering on A

Cardinal Numbers

Definition

Let A be a set. The **cardinal number** of A, denoted $\operatorname{card} A$, is the least ordinal equinumerous to A.

Cardinal Numbers

Definition

Let A be a set. The **cardinal number** of A, denoted $\operatorname{card} A$, is the least ordinal equinumerous to A.

Definition

An ordinal number is an **initial ordinal** iff it is not equinumerous to any smaller ordinal number.

Cardinal Numbers

Definition

Let A be a set. The **cardinal number** of A, denoted $\operatorname{card} A$, is the least ordinal equinumerous to A.

Definition

An ordinal number is an **initial ordinal** iff it is not equinumerous to any smaller ordinal number.

Observation

Cardinal numbers and initial ordinals are the same numbers.

Rank

Idea

We want to define hierarchy of sets indexed by ordinals:

$$V_0 = \emptyset,$$
$$V_{\alpha+1} = \mathcal{P}V_\alpha, \text{ if } \alpha \text{ is a succesor ordinal},$$
$$V_\lambda = \bigcup_{\beta<\lambda} V_\beta, \text{ if } \lambda \text{ is a limit ordinal}.$$

Regularity Axiom

Regularity (foundation) axiom

Every non-empty set A has a member m with $m \cap A = \emptyset$, that is,

$$\forall A \,[\, A \neq \emptyset \rightarrow \exists m \,(m \in A \wedge m \cap A = \emptyset)\,].$$

References

- Herbert B. Enderton (1977). Elements of Set Theory. Academic Press (cit. on pp. 2, 14).
- Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on pp. 20, 21).