CM0832 Elements of Set Theory
4. Natural Numbers

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Preliminaries

Textbook

Enderton (1977). Elements of Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Defining the Natural Numbers

Approaches for introducing mathematical objects

- Axiomatic
- Definitional

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Definitional approach for introducing natural numbers

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- We shall prove the properties of natural numbers from properties of sets.

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- We shall prove the properties of natural numbers from properties of sets.

Question

How to define natural numbers in terms of sets?

Inductive Sets

von Neumann's construction

Informal idea: A natural number is the set of all smaller natural numbers

$$\begin{aligned}
0 &:= \emptyset, \\
1 &:= \{0\} &&= \{\emptyset\}, \\
2 &:= \{0, 1\} &&= \{\emptyset, \{\emptyset\}\}, \\
3 &:= \{0, 1, 2\} &&= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\vdots
\end{aligned}$$

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&\ \vdots
\end{aligned}$$

Some 'extra' properties

$$0 \in 1 \in 2 \in 3 \in \cdots \quad \text{and} \quad 0 \subseteq 1 \subseteq 2 \subseteq 3 \subseteq \cdots .$$

Inductive Sets

A wrong impredicative definition

$$n := \{0, 1, \dots, n-1\}.$$

'We cannot just say that a set n is a natural number if its elements are all the smaller natural numbers, because such a "definition" would involve the very concept being defined.' (Hrbacek and Jech [1978] 1999, p. 40)

Inductive Sets

Definition

Let a be a set. The **successor** of a is

$$a^+ := a \cup \{a\}.$$

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Example

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \emptyset^+, \\ 2 &= \emptyset^{++}, \\ 3 &= \emptyset^{+++}, \\ &\vdots \end{aligned}$$

Inductive Sets

Definition

A set A is **inductive** iff

- $\emptyset \in A$ and
- if $a \in A$ then $a^+ \in A$.

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Question

Are there inductive sets?

Infinite Axiom

Infinity axiom

There exists an inductive set, that is,

$$\exists A\,[\,\emptyset \in A \wedge \forall a\,(a \in A \rightarrow a^{+} \in A)\,].$$

The Set of Natural Numbers

Definition

A **natural number** is a set that belongs to every inductive set.

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Theorem 4A

There is a set whose members are exactly the natural numbers.

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Theorem 4A

There is a set whose members are exactly the natural numbers.

Proof.

Let A be an inductive set. By the subset axiom scheme, there is a set

$$\{ x \in A \mid x \in I \text{ for every inductive set } I \}.$$

■

The Set of Natural Numbers

Definition

The set of **all natural numbers**, denoted by ω, is defined by

$$\omega := \{\, x \in A \mid x \in I \text{ for every inductive set } I \,\}.$$

That is,

$$x \in \omega \quad \text{iff} \quad x \text{ is a natural number.}$$

The Set of Natural Numbers

Theorem 4B

The set ω is inductive, and it is a subset of every other inductive set.

The Set of Natural Numbers

Theorem 4B

The set ω is inductive, and it is a subset of every other inductive set.

Observation

The set w is the smallest inductive set.

The Set of Natural Numbers

Observation

Since that the collection of all inductive sets is not a set but a proper class, using class we could define the set of natural numbers by

$$\omega := \bigcap \{ A \mid A \text{ is an inductive set} \}.$$

The Set of Natural Numbers

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Observation

Mendelson ([1973] 2008) in the proof of Theorem ZFC 8 defines the set ω as an intersection of some inductive sets.

Induction Principle for Natural Numbers

Induction principle for ω (p. 69)

Any inductive subset of ω coincides with ω.

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Any inductive subset of ω coincides with ω.

Induction principle for ω (other version) (Hrbacek and Jech [1978] 1999)

Let $P(x)$ be a property. Assume that

(i) $P(0)$ holds,

(ii) for all $n \in \omega$, $P(n)$ implies $P(n^+)$.

Then P holds for all natural numbers n.

Proof.

'This is an immediate consequence of our definition of w. The assumptions i) and ii) simple say that the set $A = \{ n \in \omega \mid P(n) \}$ is inductive. $\omega \subseteq A$ follows.' (Hrbacek and Jech [1978] 1999, p. 42) ∎

Defining Natural Numbers as Sets

Observation

So far, we defined natural numbers on terms of sets. A different point of view is stated by some authors (see, e.g. Benacerraf (1965)).

Induction as Foundations

'Thus inductive definibility is a notion intermediate in strength between predicate and fully impredicative definability. It would be interesting to formulate a coherent conceptual framework that made induction the principal notion. There are suggestions of this in the literature, but the possibility has not yet been fully explored.' (Aczel 1977, p. 780)

Transitive Sets

Definition

Let A be a set. The set A is a **transitive set** iff every member of a member of A is itself a member of A, that is,

$$x \in a \in A \quad \text{implies} \quad x \in A.$$

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Example

Whiteboard.

Transitive Sets

Theorem

A set A is a transitive set iff $\bigcup A \subseteq A$.

(continued on next slide)

Transitive Sets

Proof.

i) (Only if) Let A be a transitive set. Then

$$\begin{aligned} x \in \bigcup A &\Rightarrow \exists b\,(x \in b \wedge b \in A) && \text{(by definition of } \textstyle\bigcup A\text{)} \\ &\Rightarrow x \in A && \text{(because } A \text{ is transitive)} \end{aligned}$$

ii) (If) Let $\bigcup A \subseteq A$. Then

$$\begin{aligned} x \in a \wedge a \in A &\Rightarrow x \in \bigcup A && \text{(by definition of } \textstyle\bigcup A\text{)} \\ &\Rightarrow x \in A && \text{(because } \textstyle\bigcup A \subseteq A\text{)} \end{aligned}$$

■

Transitive Sets

Theorem

A set A is a transitive set iff $a \in A$ implies $a \subseteq A$.

Transitive Sets

Theorem

A set A is a transitive set iff $a \in A$ implies $a \subseteq A$.

Proof.

i) (Only if) Let A be a transitive set and let $a \in A$. If $x \in a$ implies $x \in A$ because A is transitive.

ii) (If) Let $a \in A$ implies $a \subseteq A$. If $x \in a \land a \in A$ implies $x \in A$ because $a \subseteq A$.

∎

Transitive Sets

Theorem

A set A is a transitive set iff $A \subseteq \mathcal{P}A$.

Transitive Sets

On transitive sets

Let A be a set. Transitive sets can be defined using any of the followings equivalent affirmations:

(i) $x \in a \in A$ implies $x \in A$,

(ii) $\bigcup A \subseteq A$,

(iii) $a \in A$ implies $a \subseteq A$,

(iv) $A \subseteq \mathcal{P}A$.

Transitive Sets

Theorem 4E

If a is a transitive set, then $\bigcup(a^+) = a$.

Transitive Sets

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Theorem 4F

Every natural number is a transitive set.

Transitive Sets

Theorem 4E

If a is a transitive set, then $\bigcup(a^+) = a$.

Theorem 4F

Every natural number is a transitive set.

Theorem 4G

The set ω is a transitive set.

Recursion on Natural Numbers

Recursion theorem on ω (p. 73)

Let A be a set, $a \in A$ and $F : A \to A$. Then there exists a unique function h such that

$$\begin{aligned} h &: \omega \to A \\ h(0) &= a, \\ h(n^+) &= F(h(n)), \text{ for all } n \in \omega. \end{aligned}$$

Arithmetic

Idea

We shall apply the recursion theorem to define addition and multiplication on ω.

Arithmetic

Example

We want to define the function

$$A_5 : w \to w$$
$$A_5(n) = \text{addition of } 5 \text{ to } n.$$

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$$A_5 : w \to w$$
$$A_5(n) = \text{addition of } 5 \text{ to } n.$$

Let $F : \omega \to \omega := n \mapsto n^+$. By the recursion theorem there exists a unique function

$$A_5 : w \to w$$
$$A_5(0) = 5,$$
$$A_5(n^+) = (A_5(n))^+.$$

Arithmetic

Example

Let $m \in \omega$. By the recursion theorem there exists a unique function

$$A_m : w \to w$$
$$A_m(0) = m,$$
$$A_m(n^+) = (A_m(n))^+.$$

Arithmetic

Definition

Let m and n be natural numbers. We define the **addition** of m and n by

$$(+) : w \times w \to w$$
$$m + n = A_m(n).$$

Arithmetic

Theorem 4I

Let m and n be natural numbers. Then

$$\begin{aligned} m + 0 \quad &= m, \\ m + n^+ &= (m + n)^+. \end{aligned}$$

Arithmetic

Example

Let $m \in \omega$. By the recursion theorem there exists a unique function

$$
\begin{aligned}
M_m &: w \to w \\
M_m(0) &= 0, \\
M_m(n^+) &= M_m(n) + m.
\end{aligned}
$$

Arithmetic

Definition

Let m and n be natural numbers. We define the **multiplication** of m and n by

$$\begin{aligned}(\cdot) : w \times w \to w \\ m \cdot n = M_m(n).\end{aligned}$$

Arithmetic

Theorem 4J

Let m and n be natural numbers. Then

$$\begin{aligned} m \cdot 0 \quad &= 0, \\ m \cdot n^+ &= (m \cdot n) + m. \end{aligned}$$

Ordering on Natural Numbers

Strong induction principle for ω (p. 87)

Let A be a subset of ω, and assume that for every n in ω,

$$m < n \to m \in A \quad \text{implies} \quad n \in A.$$

Then $A = \omega$.

References

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