CM0832 Elements of Set Theory
1. Introduction

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Semester 2017-2

Administrative Information

Course web page

`https://asr.github.io/courses/cm0832-set-theory/2017-2/`

Exams, bibliography, etc.

See course web page.

Textbook

Enderton (1977). Elements of Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Notation

Logical constants

$\wedge$	(and)	conjunction
$\vee$	(or)	inclusive[†] disjunction
$\rightarrow$	(if __, then __)	conditional, material implication
$\neg$	(not)	negation
$\leftrightarrow$	(if and only if)	bi-conditional, material equivalence
$\bot$	(falsity)	bottom, falsum
$\forall x$	(for every x)	universal quantifier
$\exists x$	(there exists a x)	existential quantifier
$\exists! x$	(there exists one and only one x)	unique existential quantifier
$=$	(equal)	equality, identity

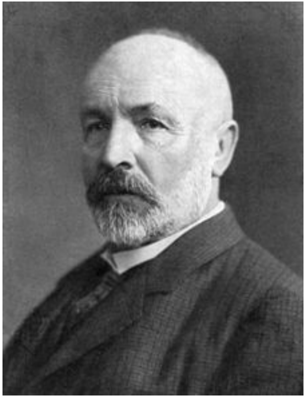
[†] One or the other or both.

Notation

Conventions

Sets will be denote by lowercase letters $(a, b, \ldots)$, uppercase letters $(A, B, \ldots)$, script letters $(\mathcal{A}, \mathcal{B}, \ldots)$ and Greek letters $(\alpha, \beta, \ldots)$.

Origins

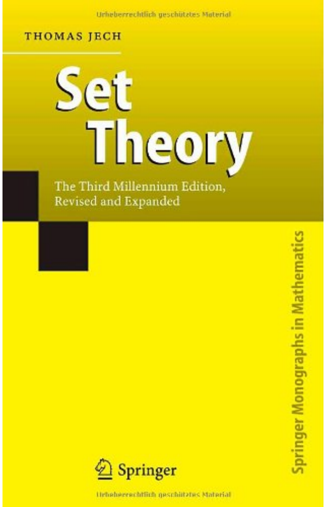


Georg Cantor (1845 – 1918)†

Cantor around 1870

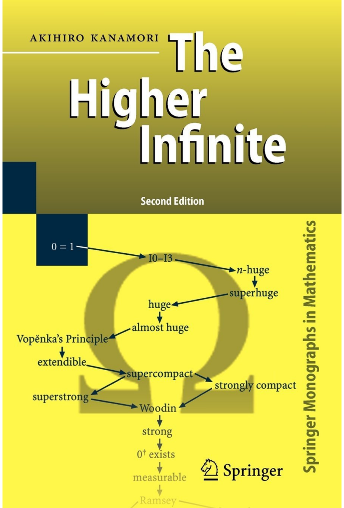
†Figures source: https://en.wikipedia.org/wiki/Georg_Cantor .

Origins



'Set theory was invented by Georg Cantor. . . It was however Cantor who realized the significance of one-to-one functions between sets and introduced the notion of cardinality of a set.' (Jech [1978] 2006, p. 15)

Origins



'Set theory was born on that December 1873 day when Cantor established that the reals are uncountable, i.e. there is no one-to-one correspondence between the reals and the natural numbers.' (Kanamori [1994] 2009, p. XII)

Naive Set Theory

Observation

Cantor set theory is also called naive set theory.

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Cantor's set definition

'By an **aggregate** *(Menge)* we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought. These objects are called the **elements** of M.' (Cantor [1915] 1955, p. 85)

Naive Set Theory

Membership relation (a binary relation)

- $t \in A$ means that t is a member of A
- $t \notin A$ means that t is not a member of A and it is defined by

$$t \notin A := \neg(t \in A).$$

Naive Set Theory

Example (Introduction to the principle of extensionality)

The first examples of sets in (Enderton 1977) are the following sets:

1. The set whose members are the prime numbers less than 10.
2. The set of all solutions to the polynomial equation

$$x^4 - 17x^3 + 101x^2 - 247x + 210 = 0.$$

Naive Set Theory

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Let's call A and B the first and the second set, respectively. Note that

$$A = \{2, 3, 5, 7\} = B.$$

Naive Set Theory

Principle of extensionality

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- $\forall A \forall B \, [\forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B \,]$.

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Principle of extensionality

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- Let A and B two sets, for all x, if $x \in A$ iff $x \in B$, then $A = B$.
- $\forall A \, \forall B \, [\, \forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B \,]$.
- Note that the converse

$$\forall A \, \forall B \, [\, A = B \rightarrow \forall x \, (x \in A \leftrightarrow x \in B) \,]$$

means something different.

Naive Set Theory

Empty set

The **empty set**, denoted by $\emptyset$, has no members.

Naive Set Theory

Empty set

The **empty set**, denoted by $\emptyset$, has no members.

- By extensionality, the set $\emptyset$ is the only set without members.
- We can define the empty set by

$$\emptyset := \{\, x \mid x \neq x \,\}.$$

Naive Set Theory

Pair set

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$$\{x, y\}$$

is the set whose only elements are x and y.

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- Note that if $x = y$ then $\{x, x\} = \{x\}$.
- Note that $\emptyset \neq \{\emptyset\}$ because $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \emptyset$.

Naive Set Theory

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Generalisation. Let $x_1, \ldots, x_n$ be objects. We can define the set

$$\{x_1, \ldots, x_n\}.$$

Naive Set Theory

Unions

Let A and B two sets, the **union** of A and B is defined by

$$A \cup B := \{ x \mid x \in A \vee x \in B \}.$$

Intersections

Let A and B two sets, the the **intersection** of A and B is defined by

$$A \cap B := \{ x \mid x \in A \wedge x \in B \}.$$

Disjoint sets

Two sets A and B are **disjoint** iff $A \cap B = \emptyset$.

Naive Set Theory

Subsets

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$$A \subseteq B := \forall x\, (x \in A \to x \in B).$$

- Note that $A \subseteq A$, for any set A.
- Note that $\emptyset \subseteq A$, for any set A.
- The membership relation and the subset relation are different (e.g. $\emptyset \subseteq \emptyset$ but $\emptyset \notin \emptyset$).

Naive Set Theory

Power set

Let A be a set. The power set of A, denoted by $\mathcal{P}A$, is the set of all subsets of A, that is,

$$\mathcal{P}A := \{\, x \mid x \subseteq A \,\}.$$

Naive Set Theory

Power set

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$$\mathcal{P}A := \{\, x \mid x \subseteq A \,\}.$$

Example

$$\mathcal{P}\emptyset = \emptyset,$$
$$\mathcal{P}\{\emptyset\} = \{\emptyset, \{\emptyset\}\},$$
$$\mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Naive Set Theory

Notation

Let A be a set. The cardinality of A is denoted by $\operatorname{card} A$.

Naive Set Theory

Notation

Let A be a set. The cardinality of A is denoted by $\operatorname{card} A$.

Theorem

If $\operatorname{card} A = n$ then $\operatorname{card}(\mathcal{P}A) = 2^n$.

Naive Set Theory

Problem

Implicit use of properties of sets.

[†] Figure source: https://commons.wikimedia.org/w/index.php?curid=48219447 .

Naive Set Theory

Problem

Implicit use of properties of sets.

An example of such properties was the axiom of choice as we shall see in the following examples.

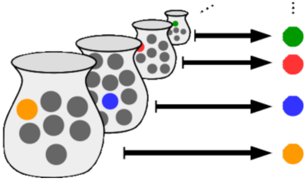


Illustration of the axiom of choice.[†]

[†]Figure source: https://commons.wikimedia.org/w/index.php?curid=48219447 .

Naive Set Theory

Example

1. Recall that a real function (i.e. a real-valued function of a real variable) f is continuous at a point p iff

$$f(x) = a \quad \text{and} \quad \lim_{x \to p} = a.$$

2. Recall also that a real function f is sequentially continuous (or Heine-continuous) at a point p iff for every sequence $\langle x_n \mid n \in \mathbb{Z}^+ \rangle$ converging to p, the sequence $\langle f(x_n) \mid n \in \mathbb{Z}^+ \rangle$ converges to $f(p)$.

3. The proof that above definitions are equivalent (Heine 1872) requires the use of axiom of choice.†

†See, e.g. (Moore 1982, p. 14) and (Hrbacek and Jech [1978] 1999, pp. 145-6).

Naive Set Theory

Example

In measure theory, the proof that a set is not Lebesgue-measurable requires the use of the axiom of choice (Solovay 1970).[†]

Video: https://www.youtube.com/watch?v=hcRZadc5KpI.[‡]

[†] See, also, (Moore 1982).

[‡] Thanks to our student Andrés Pérez-Coronado by pointing us out the video.

Naive Set Theory

Problem

Too general method of abstraction (i.e. axiom schema of unrestricted comprehension)

Naive Set Theory

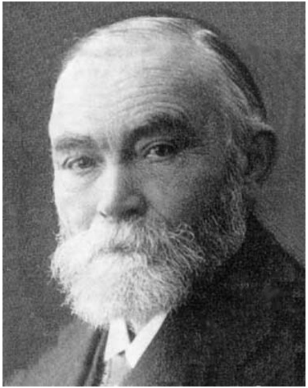
Problem

Too general method of abstraction (i.e. axiom schema of unrestricted comprehension)

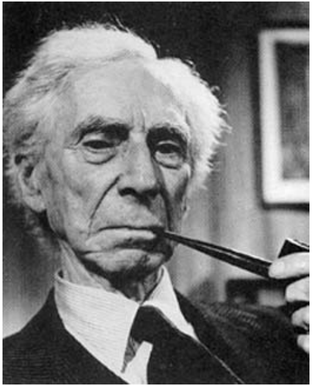
Example (Russell's paradox)

Whiteboard.

Russell's Paradox



Gottlob Frege (1848 – 1925)



Bertrand Russell (1872 – 1970)

Letter from Russell to van Heijenoort[†]

Penrhyndeudraeth, 23 November 1962
Dear Professor van Heijenoort,

As I think about acts of integrity and grace, I realise there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

Yours sincerely
Bertrand Russell

[†]van Heijenoort (1967, p. 127).

Naive Set Theory

Problem

Too general method of abstraction (i.e. axiom schema of unrestricted comprehension)

Exercise

Which is the Berry paradox?

Informally Building Sets

Definition

A set is **pure** iff its members are also sets.

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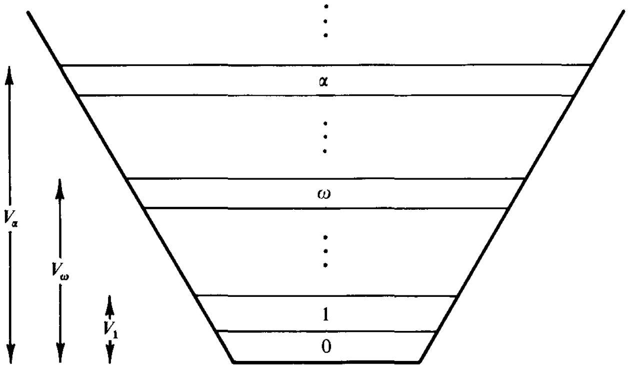
Notation

In the following two figures, ω denotes the set of natural numbers and α denotes an ordinal number greater than ω.

Informally Building Sets[†]

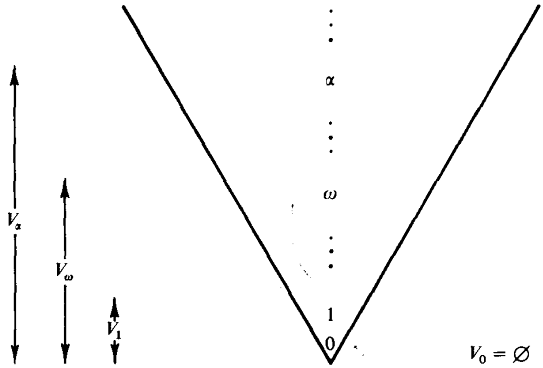


$$
\begin{aligned}
V_0 &:= A \text{ (set of atoms)} \\
V_{n+1} &:= V_n \cup \mathcal{P} V_n \\
&\vdots \\
V_\omega &:= V_0 \cup V_1 \cup \cdots \\
V_{\omega+1} &:= V_\omega \cup \mathcal{P} V_\omega \\
&\vdots \\
V_{\alpha+1} &:= V_\alpha \cup \mathcal{P} V_\alpha \\
&\vdots
\end{aligned}
$$

[†] Figure source: Enderton (1977, Fig. 2)

Informally Building Sets

The ordinal numbers are the backbone of the universe of pure sets†



$$V_0 := \emptyset \text{ (no atoms)}$$
$$V_{n+1} := \mathcal{P}V_n$$
$$\vdots$$
$$V_\omega := V_0 \cup V_1 \cup \cdots$$
$$V_{\omega+1} := \mathcal{P}V_\omega$$
$$\vdots$$
$$V_{\alpha+1} := \mathcal{P}V_\alpha$$
$$\vdots$$

† Figure source: Enderton (1977, Fig. 3)

Classes

Informal description

A set is a class, but some classes are too large to be a sets.

Example

The collection of all sets.

Observation

A class A is a set if $A \subseteq V_\alpha$ (i.e. $A \in V_{\alpha+1}$) for some ordinal number α.

Axiomatic Method

Some features

- Axioms: Explicitly list of assumptions

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- Theorems: Logical consequences of the axioms

Axiomatic Method

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- Axioms: Explicitly list of assumptions
- Theorems: Logical consequences of the axioms
- Property of set theory: It should be an axiom or a theorem

Axiomatic Method

Axiomatic set theory as a fundational system for mathematics

- 'Our axioms provide a sufficient collection of assumptions for the development of the whole of mathematics—a remarkable fact.' (Enderton 1977, p. 11)
- 'Experience has shown that practically all notions used in contemporary mathematics can be defined, and their mathematical properties derived, in this axiomatic system. In this sense, the axiomatic set theory serves as a satisfactory foundations for the other branches of mathematics.' (Hrbacek and Jech [1978] 1999, p. 3)
- 'But why axiomatize set theory in the first place? Well, for one thing, it is well known that set theory provides a unified framework for the whole of pure mathematics, and surely if anything deserves to be put on a sound basis it is such a foundational subject.' (Devlin [1979] 1993, p. 29)
- 'Conventional mathematics is based on ZFC (the Zermelo-Fraenkel axioms, including the Axiom of Choice). Working withing ZFC, on develops:. . . All the mathematics found in basic texts on analysis, topology, algebra, etc.' (Kunen [2011] 2013, p. 1)

Axiomatic Method

Some axiomatic systems

- Zermelo-Fraenkel set theory (ZF)
- Zermelo-Fraenkel set theory with Choice (ZFC)
- von Neumann-Bernays-Gödel set theory (NBG)
- Morse-Kelley set theory (MK)
- Tarski-Grothendieck set theory (TG)

Axiomatic Method

First-Order Theories†

‘The adjective “first-order” is used to distinguish the languages we shall study here from those in which there are predicates having other predicates or functions as arguments or in which predicate quantifiers or function quantifiers are permitted, or both.’ (Mendelson [1964] 2015, p. 53)

† For an introduction to first-order languages and first-order theories, see e.g. (Hamilton 1978) or (Mendelson [1964] 2015).

Axiomatic Method

Primitive notions

We only need two primitive notions, ‘set’ and ‘member’.

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Non-logical symbols

In our formalisation of ZFC, the set of non-logical symbols is

$$\mathfrak{L} = \{\epsilon\},$$

where ϵ is a binary predicate (relation) symbol.

References

- Georg Cantor [1915] (1955). Contributions to the Founding of the Theory of Transfinite Numbers. Translation of *Beiträge zur Begründung der transfiniten Mengenlehre*. Mathematische Annalen, 1895, vol. 46.2, pp. 481–512 and 1897, vol. 49.2, pp. 207–246. Translated, and provided with an introduction and notes, by Philip E. B. Jourdain. Dover Publications (cit. on pp. 8, 9).
- Keith Devlin [1979] (1993). The Joy of Sets. Fndamentals of Contemporary Set Theory. 2nd ed. Undergraduate Texts in Mathematics. Springer-Verlag (cit. on p. 46).
- Herbert B. Enderton (1977). Elements of Set Theory. Academic Press (cit. on pp. 2, 11, 12, 40, 41, 46).
- A. G. Hamilton (1978). Logic for Mathematicians. Cambridge University Press (cit. on p. 48).
- Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on pp. 31, 46).
- Thomas J. Jech [1978] (2006). Set Theory. The Third Millennium Edition, Revised and Expand. Corrected 4th printing. Springer (cit. on p. 6).

References

- Akihiro Kanamori [1994] (2009). The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings. 2nd ed. Springer (cit. on p. 7).
- Kenneth Kunen [2011] (2013). Set Theory. Revised edition. Vol. 34. Mathematical Logic and Foundations. College Publications (cit. on p. 46).
- Elliott Mendelson [1964] (2015). Introduction to Mathematical Logic. 6th ed. CRC Press (cit. on p. 48).
- Gregory H. Moore (1982). Zermelo's Axiom of Choice. Its Origins, Development, and Influence. Springer-Verlag (cit. on pp. 31, 32).
- Robert M. Solovay (1970). A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable. Annals of Mathematics 92.1, pp. 1–56. DOI: 10.2307/1970696 (cit. on p. 32).
- Jean van Heijenoort, ed. (1967). From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931. Source Books in the History of the Sciences. Harvard University Press (cit. on p. 36).